Coordinates in Operator Algebra

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We gratefully acknowledge many years of support from the National Science Foundation.

Abstract. This is a revised and expanded version of lectures I gave at TCU in 1990. The notes are still in a preliminary and changing state. Comments from friends and interested parties in general are very welcome and will be much appreciated.

Contents

Preface

This text is an embryonic preface. The full preface will be written at the end. For the time being, I will write remarks that will be included in the final version.

1. Prerequisites: Arveson's invitation to C^*-algebra [**10**].
2. Hilbert spaces will be over $\mathbb{C}$ and unless otherwise specified, separable. All operators are linear and unless specified to the contrary, bounded. If H is a Hilbert space, then $B(H)$ will denote the algebra of (bounded linear) operators on H. $K(H)$ will denote the ideal of compact operators.
3. Other separability hypotheses: All topological spaces will be assumed to be second countable and Hausdorff, unless otherwise specified. C^*-algebras will be separable, unless they appear as multiplier algebras of something else.
4. Include a brief word here about the inductive limit topology on $C_c(X)$ — X locally compact. Measures on locally compact spaces will be Radon measures, i.e., they will give finite measure to compact sets and will be regular.
5. Abstract measure spaces will usually be standard, almost always analytic, and always countably generated.
6. Image of a measure μ under a Borel map π will be $\pi(\mu)$: $\pi(\mu)(E) = \mu(\pi^{-1}(E))$

CHAPTER 1

Transformation Group C^*-algebras

1.

In this chapter we present some basic facts about transformation group C^*-algebras. Transformation groups have played a major role in operator algebra from the very beginning of the subject. Our objective here is simply to highlight special features of these algebras that have inspired the use of groupoids. Details will be given in the context of groupoids later. It is our hope that the reader will find groupoids just as manageable as transformation groups and more useful for the purposes of operator algebra.

Suppose that X is a locally compact Hausdorff space and that G is a locally compact group acting on X. We denote the action of $t \in G$ on $x \in X$ by xt; if G is abelian, we write $x+t$. The pair (X, G) will be referred to as a *transformation group*. With respect to the inductive limit topology[1], $C_c(X \times G)$ is a topological $*$-algebra when endowed with the multiplication and adjoint defined by the formulae:

$$f * g(x,t) := \int_G f(x,s)g(xs, s^{-1}t)\, ds$$
$$f^*(x,t) := \overline{f(xt, t^{-1})}.$$

Here, ds denotes left Haar measure. (See [**63**, Lemma 3.7].)

For those who are familiar with transformation group C^*-algebras, it should be noted that the modular function, which is usually included in the definition of the adjoint, has purposely been omitted. While we will have to pay for this later, in terms of other unconventional expressions, the groupoid perspective that will be developed suggests that it ought to be omitted. The map $f(x,t) \to f(x,t)\Delta^{\frac{1}{2}}(t)$, where Δ is the modular function on G, implements a $*$-isomorphism between the algebra just defined and the usual $*$-algebra structure defined on $C_c(X \times G)$.

It is instructive to reflect for a moment on the meaning of these operations. If G reduces to $\{e\}$, we have $C_c(X)$ under pointwise multiplication and conjugation, while if X reduces to a point, we have $C_c(G)$ with its usual convolution and (slightly modified) involution. What makes the multiplication interesting is the presence of the action of G on X. If there were no action, i.e., if $xt = x$ for all $t \in G$ and $x \in X$, then $C_c(X \times G)$ would be $C_c(X) \otimes C_c(G)$, where $C_c(X)$ is given its usual pointwise operations and $C_c(G)$ is endowed with its usual convolution product and involution, $f^*(t) = \overline{f(t^{-1})}$ – an algebra which, from the perspective of operator algebra, is essentially no more complicated than the two factors, $C_c(X)$ and $C_c(G)$.

[1] Let Y be a locally compact Hausdorff space and for each compact subset K of Y, let $C_c(Y)^K$ be the set of all functions in $C_c(Y)$ that are supported on K, endowed with the supremum norm. Then the inductive limit topology on $C_c(Y)$ is the unique largest locally convex topology with the property that for each compact subset K of Y, the identity injection, $i_K : C_c(Y)^K \to C_c(Y)$, is continuous. (See [**70**, II.14.3].)

When the action of G on X is not trivial, the algebra $C_c(X \times G)$ becomes quite interesting. Furthermore, what makes these algebras attractive is that they exhibit interesting operator algebraic phenomena even when the groups and spaces are not particularly complicated.

Two basic and closely related examples of transformation groups will serve many of our needs in these notes. They are the source of inspiration for much of the current work in operator algebra.

EXAMPLE 1.1. 1. *We view the 1-torus, or the unit circle,* $\mathbb{T}$, *as* $\mathbb{R}/\mathbb{Z}$ *and write the image of* $\theta \in \mathbb{R}$ *in* $\mathbb{T}$ *as* $[\theta]$. *Given* $\alpha \in \mathbb{R}$, *the integers* $\mathbb{Z}$ *may act on* $\mathbb{T}$ *according to the formula* $[\theta] + n = [\theta + \alpha n]$.

2. *We also view* $\mathbb{T}^2$ *as* $\mathbb{R}^2/\mathbb{Z}^2$ *and write the elements in* $\mathbb{T}^2$ *as* $[\theta, \eta]$, $(\theta, \eta) \in \mathbb{R}^2$. *For* $\alpha \in \mathbb{R}$, *we let* $\mathbb{R}$ *act on* $\mathbb{T}^2$ *according to the formula* $[\theta, \eta] + t = [\theta + t, \eta + \alpha t]$.

The completion of $C_c(X \times G)$ that we wish to study is constructed through representations that we now define.

DEFINITION 1.2. *A* representation *of* $C_c(X \times G)$ *is simply a* $*$-*homomorphism,*

$$\pi : C_c(X \times G) \to B(H),$$

from $C_c(X \times G)$ *to* $B(H)$, *for some Hilbert space* H, *that is continuous with respect to the inductive limit topology on* $C_c(X \times G)$ *and the weak operator topology on* $B(H)$ *and that satisfies the nondegeneracy condition:*

$$\{\pi(f)\xi \mid f \in C_c(X \times G),\ \xi \in H\} \text{ is dense in } H.$$

We note that the nondegeneracy condition appears in subtle ways in arguments and cannot be ignored altogether, as one might think at first glance.

DEFINITION 1.3. *For* $f \in C_c(X \times G)$, $\|f\|$ *is defined to be*

$$\sup\{\|\pi(f)\| \mid \pi \text{ — a representation of } C_c(X \times G)\}.$$

One asks immediately: Is this quantity finite? Is it a norm? The supremum is finite and it does define a norm, but to prove this requires some preparation. Before discussing it, however, we state for reference

PROPOSITION 1.4. *The quantity,* $\|\cdot\|$, *is a* C^*-*norm on* $C_c(X \times G)$ *and the completion of* $C_c(X \times G)$ *with respect to it is a* C^*-*algebra. We call this* C^*-*algebra the* transformation group C^*-algebra *associated with the transformation group* (X, G) *and denote it by* $C^*(X, G)$.

To prove 1.4, we need to know how to construct representations of $C_c(X \times G)$. The key ingredient is defined in

DEFINITION 1.5. *A* covariant representation *of* (X, G) *consists of a pair* (ρ, U), *where* $\rho : C_0(X) \to B(H)$ *is a nondegenerate* C^*-*representation and* $U : G \to \mathcal{U}(H)$ *— the unitary group in* $B(H)$ *— is a strongly continuous unitary representation such that*

$$U(t)\rho(\varphi)U(t^{-1}) = \rho(\varphi_t), \tag{1.1}$$

$\varphi \in C_0(X)$, *where* $\varphi_t(x) = \varphi(xt)$.

The equation (1.1) is called the *covariance condition.* Sometimes a covariant representation is also called a *system of imprimitivity*.

Given a covariant representation (ρ, U), we define $\rho \times U : C_c(X \times G) \to B(H)$ by the formula $\rho \times U(f) := \int_G \rho(f(\cdot, t))U(t)\,dt,\ f \in C_c(X \times G)$. The integral is calculated in the weak or strong operator topology. A moment's reflection reveals that $\rho \times U$ is a representation of $C_c(X \times G)$.

THEOREM 1.6. **[63]** *The map* $(\rho, U) \to \rho \times U$ *is a* bijection *between the covariant representations of* (X, G) *and the representations of* $C_c(X, G)$. *That is, every representation* π *of* $C_c(X \times G)$ *may be written as* $\pi = \rho \times U$ *for a unique covariant representation* (ρ, U).

The representation $\rho \times U$ associated with a covariant representation (ρ, U) is sometimes called the *integrated form* of (ρ, U), while the covariant representation associated with a representation π is called the *disintegrated form* of π. We will not prove Theorem 1.6 here; it will fall out of later material. Indeed, Renault's disintegration theorem, Theorem 3.32, which is the goal of Chapter 3, may be viewed as a far-reaching generalization of this theorem. However, we do want to note the key point of the proof: Given a representation π of $C_c(X \times G)$, it is natural to try to define $U(t)\pi(f)\xi = \pi(f_t)\xi$, where $f_t(x, s) := f(xt, t^{-1}s)$, and to define $\rho(\varphi)\pi(f)\xi = \pi(\varphi \cdot f)\xi$, where $\varphi \cdot f(x, t) := \varphi(x)f(x, t)$. The problem is first to show that U and ρ are well-defined, and then that $\pi = \rho \times U$. For these purposes, an important ingredient is an approximate identity:

$$e_n := \varphi_n \cdot \delta_n,$$

where $\{\varphi_n\}$ is an approximate identity for $C_c(X)$, $\{\delta_n\}$ is an approximate identity for $C_c(G)$, and $\varphi_n \cdot \delta_n(x, t) = \varphi_n(x)\delta_n(t)$. As we shall see in the proof of Renault's disintegration theorem, Theorem 3.32, and elsewhere in the theory, approximate identities play an important role in the study of groupoids.

Given Theorem 1.6, Proposition 1.4 is almost immediate. Indeed, given a representation π of $C_c(X \times G)$, written $\pi = \rho \times U$, we have

$$\|\pi(f)\| \leq \int \|f(\cdot, t)\|_\infty dt := \|f\|_0, \tag{1.2}$$

$f \in C_c(X \times G)$, because

$$\begin{aligned} |(\rho \times U(f)\xi, \eta)| &= \left| \int_G (\rho(f(\cdot, t))U(t)\xi, \eta)\,dt \right| \\ &\leq \int_G \|\rho(f(\cdot, t))\|_\infty \,|(U(t)\xi, \eta)|\,dt \\ &\leq \int_G \|f(\cdot, t)\|_\infty\,dt\,\|\xi\|\,\|\eta\| = \|f\|_0\|\xi\|\|\eta\|. \end{aligned}$$

Thus $\|f\| \leq \|f\|_0$, which is clearly finite for each $f \in C_c(X \times G)$. The rest of the proof requires no further comment.

To proceed further with the theory, we need to analyze more carefully the structure of the covariant representations. However, before doing this, let us discuss some facts that make the theory of transformation group C^*-algebras particularly piquant. Proofs of most of them will be given in the context of groupoid C^*-algebras, in later chapters. First of all, it is easy to see that $C^*(X, G)$ is unital precisely when X is compact and G is discrete. In this case, $1_{X \times \{e\}}$ serves as the identity[2]. If G acts *freely* on X in the sense that the equation $xt = x$ implies $t = e$, and if G is discrete and amenable, then elements in $C^*(X, G)$ may be viewed as functions in $C_0(X \times G)$. Otherwise, elements in $C^*(X, G)$ may be "far removed" from functions on $X \times G$. This is analogous to what happens in group C^*-algebras, even algebras as simple as $C^*(\mathbb{R})$. Of course $\mathbb{R}$ is amenable, but not discrete. In this case $C^*(\mathbb{R})$ is naturally isomorphic to $C_0(\hat{\mathbb{R}})$ via the Fourier transform. The space $\hat{\mathbb{R}}$ is naturally isomorphic to $\mathbb{R}$, indeed, it is always identified with $\mathbb{R}$, but it is important to think of it as a distinct copy of $\mathbb{R}$. However, strictly speaking, $C^*(\mathbb{R})$ is a space of distributions on $\mathbb{R}$, some of which are *not* functions, whose distributional Fourier transforms are continuous functions on $\hat{\mathbb{R}}$ that vanish at infinity.

If $U \subseteq X$ is open and invariant, then $C^*(U, G)$ may be viewed as an ideal in $C^*(X, G)$ in an obvious way: Simply extend functions in $C_c(U \times G)$ to functions in $C_c(X \times G)$ by declaring them to be zero outside $U \times G$ and take the closure in $C^*(X, G)$. If G acts freely on X and if G is amenable, then the map $U \to C^*(U, G)$ from open invariant subsets of X to ideals in $C^*(X, G)$ is a bijection. The fact that the map is an injection is easy to see. The fact that it is surjective is much deeper. It is a consequence of Gootman and Rosenberg's solution [**79**] to the so-called Effros-Hahn conjecture [**63**] which deals with the ideal structure of $C^*(X, G)$ without the assumption of free action. This has been generalized to groupoid C^*-algebras by Renault [**175**] and we will discuss his proof in Chapter 6.

In general, the ideal theory of $C^*(X, G)$ depends upon how G acts on X and how the *isotropy* or *stability groups* vary. (The isotropy group at x is defined to be $\{t \in G \mid xt = x\}$.) The analysis of group actions and the "distribution of isotropy" is an active area research today. Observe that if $X = \mathbb{T}$, with $\mathbb{Z}$ acting as in Example 1.1, the action is free if and only if α is irrational. In this case $C^*(X, \mathbb{Z})$ is simple. On the other hand, if α is rational and expressed in lowest terms as $\frac{m}{n}$, then $C^*(\mathbb{T}, \mathbb{Z})$ is naturally isomorphic to $C(\mathbb{T}/\mathbb{Z}) \otimes M_n(\mathbb{C})$, where $\mathbb{T}/\mathbb{Z}$ denotes the quotient space. This statement will be easy to prove after some general preparation, but the reader is encouraged to try to prove it now to become acclimated to the thinking that will drive later developments.

We turn next to some examples and the analysis of some specific covariant representations.

EXAMPLE 1.7. *Let μ be a (positive) measure on X, let λ be left Haar measure on G, and form the Hilbert space $H = L^2(X \times G, \mu \times \lambda)$. We define* $\operatorname{Ind}\mu$ *to be $\rho \times U$ where $\rho(\varphi)\xi(x, s) = \varphi(xs)\xi(x, s)$, $\varphi \in C_0(X)$, and $U(t)\xi(x, s) = \xi(x, st)\Delta^{\frac{1}{2}}(t)$ (recall, Δ is the modular function of G). As the notation suggests,* $\operatorname{Ind}\mu$ *is called the representation of $C^*(X, G)$ induced by μ. To be more precise, we should say that* $\operatorname{Ind}\mu$ *is the representation induced by the multiplicity free representation of $C_0(X)$ determined by letting $C_0(X)$ act via multiplication on $L^2(X, \mu)$. We will have occasion to see representations of $C^*(X, G)$ induced by more general representations of $C_0(X)$ later in the context of groupoid C^*-algebras.*

PROPOSITION 1.8. *If the action is free at x, i.e., if $xt = x \Rightarrow t = e$, then* $\operatorname{Ind}\varepsilon_x$ *is irreducible.*[3]

PROOF. Since the action is assumed to be free at x, the map $t \to xt$ is a Borel isomorphism between G, viewed as a standard Borel space, and the orbit

[2]We use the notation 1_E to denote the characteristic function of a set E.

[3]Throughout, we use ε_x to denote the point mass at x.

of x with its relative Borel structure. This map implements a unitary equivalence between $\operatorname{Ind}\epsilon_x$ and $\rho_x \times U'$, where U' is the right regular representation of G on $L^2(G)$ and ρ_x is the representation of $C_0(X)$ on $L^2(G)$ defined by the formula $\rho_x(\varphi)\xi(s) = \varphi(xs)\xi(s) := \varphi_x(s)\xi(s)$. The freeness of the action at x guarantees that the functions φ_x, $\varphi \in C_0(X)$, separate the points of G. Consequently, $\rho_x(C_0(X))''$ is $L^\infty(G)$, viewed as multiplication operators on $L^2(G)$. Since this is a masa in $B(L^2(G))$ (i.e., a maximal abelian selfadjoint algebra of operators on $L^2(G)$) with no invariant projections for U', it follows easily that $\operatorname{Ind}\epsilon_x$ is irreducible. □

Given $\varphi \in L^\infty(X,\mu)$, define $\pi(\varphi)\xi(x,s) = \varphi(x)\xi(x,s)$. Then $\pi(\varphi)$ commutes with $\operatorname{Ind}\mu$. In fact, this implies that $\operatorname{Ind}\mu$ is the direct integral of all the $\operatorname{Ind}\epsilon_x$, $\operatorname{Ind}\mu \cong \int_X^\oplus \operatorname{Ind}\epsilon_x\, d\mu(x)$. If the action is free on all of X (or at least on the support of μ) $\pi(L^\infty(X,\mu))$ is a masa in the commutant of $\operatorname{Ind}\mu(C^*(X,G))$. Thus, in the presence of free action, we have a natural decomposition of $\operatorname{Ind}\mu$ into a direct integral of irreducible representations. It should be emphasized that in general this decomposition is not unique in any sense. Other choice of masas in the commutant of $\operatorname{Ind}\mu(C^*(X,G))$ may lead to decompositions of $\operatorname{Ind}\mu$ into irreducible representations of $C^*(X,G)$ that are quite different from $\operatorname{Ind}\varepsilon_x$. The decomposition of $\operatorname{Ind}\mu$ is unique (for every measure μ) precisely when $C^*(X,G)$ is a type I, or postliminal, C^*-algebra. This, in turn, rests delicately on the orbit structure of X. We will find this matter surfacing again and again in our discussion; it is a central theme in the theory of groupoid C^*-algebras.

Remark 1.9. *To pursue this theme and for other purposes as well, we begin by noting that the following computation shows that* $\operatorname{Ind}\mu$ *is rather like a right regular representation. It leads one to speculate about a left regular representation and possible relations between the two.*

$$\begin{aligned}
\operatorname{Ind}\mu(f)\xi(x,s) &= \left(\int \rho(f(\cdot,t))U(t)\xi\, d\lambda(t)\right)(x,s)\\
&= \int f(xs,t)\xi(x,st)\Delta^{\frac12}(t)\, d\lambda(t)\\
&= \int f(xs,s^{-1}r)\Delta^{\frac12}(s^{-1}t)\xi(x,t)\, d\lambda(t)\\
&= \int \xi(x,t)g^+(xt,t^{-1}s)\, d\lambda(t)\\
&= \xi * g^+(x,s),
\end{aligned}$$

where $f \in C_c(X\times G)$, $g(x,s) = f(x,s)\Delta^{\frac12}(s)$, *and* $g^+(x,s) = g(xs,s^{-1})$.

Definition 1.10. *A measure* μ *on* X *is called* quasi-invariant (q.i.) *in case* μ_t *is mutually absolutely continuous with respect to* μ, *where* μ_t *is the translate of* μ *by* t*; i.e.,* $\int f(x)d\mu_t = \int f(xt)d\mu$, $f \in C_c(X)$.

Let μ be quasi-invariant and set $J_0(\cdot,t)$ equal to the Radon-Nikodym derivative $d\mu_t/d\mu$. Note that J_0 is defined only up to a μ-null set that, a priori, depends on t. Moreover, an application of the chain rule shows that for each s and t there is a μ-null set N, depending on s and t, such that

$$J_0(x,st) = J_0(x,s)J_0(xs,t) \tag{1.3}$$

for all $x \notin N$. The question immediately arises: Is it possible to choose J_0 so that equation (1.3) is satisfied for all x, s and t without exception *and* so that J_0 is a Borel function on $X \times G$? This question has a long history with variously qualified affirmative answers. It is the sort of problem that appears often in representation theory and in the theory of stochastic processes.

From the perspective of these notes, the most satisfactory answer is due Arlan Ramsay [**157, 161**] and will be presented in Chapter 4. We state here for later reference a special case of Ramsay's theorem which gives an affirmative answer to the question in the specific setting in which it is posed. A proof of the special case, requiring very little technology, may be found in Appendix B of Zimmer's book [**205**]. However, it contains the key idea of Ramsay's theorem [**157**, Theorem 5.1] which may be traced back to Mackey's Lemma 6.2 in [**118**].

In order to present it, we require a bit more terminology. Suppose that H is a Borel group, meaning that H has a Borel structure in which the group operations are Borel. Suppose, too, that μ is a σ-finite measure on X which is quasi-invariant under the action of G.

DEFINITION 1.11. i) *A Borel function $C : X \times G \to H$ is called a (1-)* cocycle *provided that for each $s, t \in G$, there is a μ-null set $N \subseteq X$, possibly depending upon s and t, such that*

$$C(x, st) = C(x, s)C(xs, t) \tag{1.4}$$

for all $x \notin N$.

ii) *If C satisfies this equation for all x, s and t, without exception, then C is called a* strict *cocycle.*

iii) *The cocycle C is called a* coboundary *in case there is a Borel function $B : X \to H$ such that for each $t \in G$, $C(x,t) = B(x)B(xt)^{-1}$ a.e. μ. Again the exceptional null set may depend upon t. The function B, itself, is often called a coboundary.*

iv) *Given two cocycles, C_1 and C_2 on $X \times G$ with values in H, we say that C_1 and C_2 are* cohomologous *(or* equivalent *or* similar*) in case there is a coboundary B such that*

$$C_2(x, t) = B(x)C_1(x, t)B(xt)^{-1} \tag{1.5}$$

a.e. μ, for each $t \in G$.

v) *If, in iv), B can be chosen so that the equation is satisfied for all x and t, then C_1 and C_2 are called* strictly *cohomologous.*

Observe that equation (1.3) is the assertion that J_0 is a cocycle with values in the multiplicative group of positive real numbers. Our question asks if J_0 can be replaced by a strict version. Observe, too, that μ is equivalent to a σ-finite *invariant* measure precisely when J_0 is a coboundary.

THEOREM 1.12. *(cf.* [**205**, Theorem B.9]*) Suppose that H is a topological group whose Borel structure is countably generated and suppose that $C_0 : X \times G \to H$ is a cocycle. Then there is a strict cocycle $C : X \times G \to H$ such that for each $t \in G$, $C_0(x, t) = C(x, t)$ a.e. μ.*

One can prove, too, that cohomologous cocycles may be represented by strict cocycles in such a way that the cobounding equation is satisfied on all of $X \times G$.

From now on, when discussing a quasi-invariant measure, we will write J for a strict solution to the cocycle equation (1.3). The notion of quasi-invariance for measures is an important generalization of invariance. Many of the ideas that make sense for invariant measures extend to quasi-invariant measures with little or no difficulty. Moreover, as we shall see, from some perspectives quasi-invariance is an easier concept to deal with than invariance. A quasi-invariant measure is called *ergodic* if the only invariant Borel sets are either null or conull (i.e., the complement is null). Just as an invariant measure can be "decomposed" into *ergodic* pieces, so can a quasi-invariant measure. This has an operator algebraic flavor which we shall formulate below.

A *measure class* is a family C of measures such that any two measures in C are mutually absolutely continuous. We say that a measure class C is *invariant* in case $\mu_t \in C$ for all $t \in G$ whenever $\mu \in C$. Thus C is invariant if and only if each $\mu \in C$ is quasi-invariant. An invariant measure class is called *ergodic* if and only if each μ in C is ergodic.

Recall that a representation ρ of $C_0(X)$ determines a unique measure class on X. Indeed, through the Riesz representation theorem, each vector ξ in the Hilbert space of ρ gives rise to a measure μ_ξ on X defined by the formula

$$(\rho(\varphi)\xi, \xi) = \int \varphi \, d\mu_\xi \quad .$$

It is easy to see that two separating vectors[4] for $\rho(C_0(X))''$ yield mutually absolutely continuous measures on X. The measure class that these determine is called the *measure class of* ρ. Observe that if (ρ, U) is a covariant representation of (X, G), then the measure class of ρ is invariant. (This is because a unitary operator that normalizes a von Neumann algebra carries separating vectors to separating vectors.) We will call $\pi = \rho \times U$ *ergodic* if the measure class of ρ is ergodic. Observe that if E is an invariant Borel set, then $\tilde{\rho}(1_E)$ lies in the center of $\pi(C^*(X,G))''$, where $\tilde{\rho}$ is the canonical extension of ρ to $L^\infty(X, \mu)$ and where μ is any measure in the measure class of ρ. This leads one to focus attention on ergodic representations, since one may disintegrate a representation π over the center of $\pi(C^*(X,G))''$.

THEOREM 1.13. *If $\pi = \rho \times U$ is a representation of $C^*(X,G)$, then the measure class of ρ is invariant and there is a direct integral decomposition $\pi = \int_Y^\oplus \pi_y \, d\nu(y)$ where each π_y is ergodic.*

It should be noted that the decomposition $\pi = \int_Y^\oplus \pi_y \, d\nu(y)$ is not necessarily the central decomposition of π (i.e., π_y need not be factorial for y in a set of positive measure), but in a sense, it is a good start. Furthermore, the description of an ergodic representation can be carried out in a bit simpler fashion than that of an arbitrary representation. This is the implication of the following theorem.

THEOREM 1.14. *(See* **[118]** *and* [**157**, Section 10].*) Let $\pi = \rho \times U$ be an ergodic representation of $C^*(X,G)$ on a Hilbert space H. Let μ be a quasi-invariant measure in the measure class of ρ and let J be a strict cocycle satisfying $J(\cdot, t) = d\mu_t/d\mu$. Then there exist (i) a Hilbert space H_0, (ii) a strict cocycle*

$$\Theta : X \times G \to \mathcal{U}(H_0),$$

[4]Recall that a vector ξ in a Hilbert space is called a separating vector for an algebra A of operators if the equation $a\xi = 0$, $a \in A$, implies $a = 0$. Commutative von Neumann algebras on separable Hilbert spaces always have separating vectors: Simply let ξ be the sum of a sequence of vectors, properly weighted, whose complex linear span is the whole space.

with values in the unitary group of H_0, $\mathcal{U}(H_0)$, *endowed with the strong operator topology, and (iii) a Hilbert space isomorphism* $W : H \longrightarrow L^2(X, \mu, H_0)$ *such that* $(W\rho(\varphi)W^{-1}\xi)(x) = \varphi(x)\xi(x)$, $\xi \in L^2(X, \mu, \xi)$ *and* $(WU(t)W^{-1}\xi)(x) = \Theta(x, t)\xi(xt)J^{\frac{1}{2}}(xt, t^{-1})$.

OUTLINE OF PROOF. First diagonalize ρ, writing $H = \int_X^{\oplus} H(x)\, d\mu(x)$ and $[\rho(\varphi)\xi](x) = \varphi(x)\xi(x)$. The measure μ is assumed to be quasi-invariant and ergodic, and this implies that the dimension of $H(x)$ is constant almost everywhere. Therefore, we may write $H = L^2(X, \mu, H_0)$ for some fixed Hilbert space H_0. Define $\{V(t)\}_{t \epsilon G}$ on H by the formula $(V(t)\xi)(x) = \xi(xt)J^{\frac{1}{2}}(xt, t^{-1})$. It is easy to check that $\{V(t)\}_{t\epsilon G}$ is a unitary representation and that (ρ, V) is a covariant representation. ($\{V(t)\}_{t\epsilon G}$ is called the *permutation representation* of G determined by μ and H.) Consequently, $U(t)V(t)^{-1} = U(t)V(t^{-1})$ commutes with ρ, and so is a decomposable operator. Hence, for each $t \in G$, there is a Borel function $\Theta(\cdot, t) : X \to \mathcal{U}(H_0)$ such that $(U(t)V(t)^{-1}\xi) = \Theta(x, t)\xi(x)$ for almost all x. Rewriting this equation yields $(U(t)\xi)(x) = \Theta(x, t)\xi(xt)J^{\frac{1}{2}}(xt, t^{-1})$. As in our discussion about J, $\Theta(\cdot, t)$ is determined only up to a null set that depends upon t and it is not evident, a priori, that Θ is a Borel function of the two variables, x and t. However, the strong continuity of $\{U(t)\}_{t\epsilon G}$ and $\{V(t)\}_{t\epsilon G}$ implies that Θ may be chosen to be Borel. Furthermore, the group property of $\{U_t\}_{t\epsilon G}$ implies that Θ is a cocycle. Theorem 1.12 may then be applied to replace Θ by a strict version, once it is noted that $\mathcal{U}(H_0)$ satisfies the hypothesis placed on the group H in it. □

THEOREM 1.15. *(See* [**118**].*) For* $i = 1, 2$, *let* π_i *be an ergodic representation of* $C^*(X, G)$ *and let* $(\mu_i, H_{0i}, \Theta_i)$ *be the data associated to* π_i *in Theorem 1.14. Then* π_1 *is unitarily equivalent to* π_2 *if and only if* μ_1 *is equivalent to* μ_2, $\dim H_{01} = \dim H_{02}$ *and, after identifying* μ_1 *with* μ_2 *and* H_{01} *with* H_{02}, Θ_1 *and* Θ_2 *are cohomologous.*

OUTLINE OF PROOF. If the conditions are satisfied and if B is a coboundary connecting $\Theta_1$and Θ_2, then B induces a decomposable unitary operator on $L^2(\mu, H_0)$ ($\mu = \mu_1 = \mu_2$, $H_0 = H_{01} = H_{02}$) implementing a unitary equivalence between π_1 and π_2. Conversely, if π_1 and π_2 are unitarily equivalent, say by W, then so are ρ_1 and ρ_2, where $\pi_i = \rho_i \times U_i$. Standard multiplicity theory enables one to conclude that $\mu_1 \sim \mu_2$ and that $\dim H_{01} = \dim H_{02}$, and after identifications are made, that W is decomposable. If W is given by multiplication by a unitary-valued function B, then the equation $WU_1(t)W^{-1} = U_2(t)$ readily implies that B is a coboundary connecting Θ_1 and Θ_2. □

EXERCISE 1.16. *If* ν *is a positive measure on* X *and if* $\pi = \operatorname{Ind}\nu$, *when is* π *ergodic and what are* μ, H_0 *and* Θ*?*

This exercise is nontrivial without additional preparation, but reflection now on how to deal with it should prove illuminating. (See Example 3.28.) In this connection, it is worthwhile to address a special case.

Suppose ν is quasi-invariant. In this event, in Theorem 1.14, we take $\mu = \nu$, we let $H_0 = L^2(G)$, we identify $L^2(X \times G, \nu \times \lambda)$ with $L^2(X, \nu, L^2(G))$, and we let $(\Theta(\cdot, t)\xi)(x, s) = \xi(x, t^{-1}s)$. Define $W : L^2(X \times G, \nu \times \lambda) \to L^2(X \times G, \nu \times \lambda)$ by

$$(W\xi)(x, s) = J^{\frac{1}{2}}(xs, s^{-1})\Delta^{\frac{1}{2}}(s^{-1})\xi(xs, s^{-1}). \tag{1.6}$$

Then a computation shows that

$$WU(t)W^{-1}\xi(x, s) = \xi(xt, t^{-1}s)J^{\frac{1}{2}}(xt, t^{-1}) = (\Theta(\cdot, t)V(t)\xi)(x, s),$$

where

$$V(t)\xi(x,s) = \xi(xt,s)J^{\frac{1}{2}}(xt,t^{-1}),$$

and

$$W\rho(\varphi)W^{-1}\xi(x,s) = \varphi(x)\xi(x,s).$$

Thus W transforms the covariant pair (ρ, U) for $\operatorname{Ind}\nu$ into the canonical form of Theorem 1.14. Moreover, as a by-product, we see that for f in $C_c(X \times G)$,

$$\begin{aligned} W(\operatorname{Ind}\nu(f))W^{-1}\xi(x,s) &= \int f(x,t)J^{\frac{1}{2}}(xt,t^{-1})\xi(xt,t^{-1}s)\,ds \\ &= \tilde{f} * \xi(x,s), \\ \tilde{f}(x,t) &= f(x,t)J^{\frac{1}{2}}(xt,t^{-1}). \end{aligned}$$

Thus, while $\tilde{f}$ need not be in $C_c(X \times G)$, it is easily deduced from this computation that the commutant of $\operatorname{Ind}\nu(C^*(X,G))$ is transformed by W into the weak closure of $\operatorname{Ind}\nu(C^*(X,G))$. This, really, is the meaning of quasi-invariance for our purposes, and these calculations help to clarify the speculation suggested in Remark 1.9.

We have commented on how the structure of the open invariant subsets of X is reflected in the ideal structure of $C^*(X,G)$. In a sense, the decomposition of X into invariant Borel sets gives a finer decomposition of $C^*(X,G)$. After all, when the action is free, two ergodic representations of $C^*(X,G)$ are either disjoint or weakly equivalent (we define and discuss these notions thoroughly in Chapter 6). The first case occurs when the associated quasi- invariant measures are supported on disjoint, invariant Borel sets, and the second occurs when the measures are equivalent. It is natural to reflect, then, on what happens when there is precisely one orbit in X, i.e., when (X,G) is *transitive*. This, historically, was the first situation to be considered and the analysis to follow is due to Mackey [**114, 115, 118**].

Suppose (X,G) is transitive, so that for each x and y in X, there is a $t \in G$ such that $xt = y$. Pick x_0 and let $H = \{t \in G \mid x_0 t = x_0\}$ be the isotropy group at x_0. Then H is a closed subgroup of G and the map $\varphi : Ht \mapsto x_0 t$ from the right coset space $H\backslash G$ to X is a well-defined homeomorphism that is equivariant for the natural action of G on $H\backslash G$ and the action of G on X. (That is, $\varphi(Hts) = \varphi(Ht)s$, for all s and t.) Henceforth, we will identify $(H\backslash G, G)$ with (X,G).

Our global separability assumptions enable us appeal to [**115**, Lemma 1.1] to find a Borel map $\gamma : H\backslash G \to G$ so that $\pi(\gamma(Ht)) = Ht$ and $\gamma(H) = e$. Here, π is the quotient map, $\pi(t) = Ht$. In general, it is *not* possible to choose γ to be continuous. For if so, G would be homeomorphic to $H \times H\backslash G$. (Contemplate $G = \mathbb{R}$ and $H = \mathbb{Z}$).

THEOREM 1.17. [**115**] *There is a* unique *invariant measure class on $H\backslash G$, and it is ergodic.*

Indeed, simply take a probability measure on G that is equivalent to Haar measure and take the class of $\pi(\mu)$.

PROPOSITION 1.18. [**115, 118**] *There is a bijection between unitary equivalence classes of representations of H and strict cohomology classes of strict cocycles on $H\backslash G \times G$.*

PROOF. The correspondence is quite explicit. Given a strict cocycle $\Theta : H\backslash G \times G \to \mathcal{U}(K_0)$, define $L : H \to \mathcal{U}(K_0)$ by the formula $L(t) = \Theta(H, t)$. That L is a unitary representation of H is a calculation. That it is strongly continuous results from the fact that Θ is Borel. For the converse, first note that for $s, t \in G$, $\gamma(Ht)s\gamma(Hts)^{-1}$ lies in H. Indeed, since $\gamma(Ht) \in Ht$, we may write $\gamma(Ht) = h_t t$. We may also arrange to have $h_e = e$. Then $\gamma(Ht)s\gamma(Hts)^{-1} = (h_t t)s(h_{ts}ts)^{-1} = h_t h_{ts}^{-1}$. Given a unitary representation L of H on the Hilbert space K_0, simply define

$$\Theta_L(Ht, s) = L[\gamma(Ht)s\gamma(Ht)^{-1}] = L(h_t h_{ts}^{-1}).$$

Then it is easy to check that Θ_L is a strict cocycle:

$$\Theta_L(Ht, s_1 s_2) = L(h_t\, h_{ts_1 s_2}^{-1}),$$

while

$$\Theta_L(Ht, s_1)\Theta_L(Hts_1, s_2) = L(h_t\, h_{ts_1}^{-1})L(h_{ts_1}\, h_{ts_1 s_2}^{-1}).$$

Moreover, given $\Theta_i,\ i = 1, 2$, with

$$W(\Theta_1(H, t))W^{-1} = \Theta_2(H, t), \qquad t \in H,$$

for some unitary operator W (i.e., assume the associated representations of H are unitarily equivalent), then $B(Ht)$, defined to be $W\Theta_1(H, h_t)$, is a coboundary implementing a similarity between Θ_1 and Θ_2. Conversely, if Θ_1 and Θ_2 are strictly cohomologous cocycles, made so by the coboundary B, then the unitary representations of H, $\Theta_1(H, \cdot)$ and $\Theta_2(H, \cdot)$, are unitarily equivalent via $B(H)$. □

DEFINITION 1.19. *Let $X = H\backslash G$, let μ be a quasi-invariant measure on X and let $L : H \to \mathcal{U}(K_0)$ be a unitary representation of H. Define $U^L : G \to B(L^2(X, \mu, K_0))$ by the equation*

$$(U^L(t)\xi)(Hs) = \Theta_L(Hs, t)\xi(Hst)J^{\frac{1}{2}}(Hs, t).$$

Then U^L is called the unitary representation of G induced *by L.*

Give a proof in Chapter 5, after Remarks 5.41.

THEOREM 1.20. *(Mackey's Imprimitivity Theorem* **[114]**. *See* **[118]**, *also.) The map which sends a unitary representation L of H to the representation $\rho \times U^L$ of the transformation group C^*-algebra $C^*(H\backslash G, G)$, where ρ is the representation of $C_0(G/H)$ by multiplication operators on $L^2(G/H, \mu, K_0)$, preserves unitary equivalence and determines a bijection between unitary equivalence classes of unitary representations of H and unitary equivalence classes of representations of $C^*(H\backslash G, G)$. This correspondence further preserves reducibility and establishes an isomorphism between the commutant of L and the commutant of $\rho \times U^L$.*

The proof is easily assembled from what has been presented and so will be left to the reader.

If (X, G) is *not* transitive then a dichotomy arises, either every ergodic quasi-invariant measure is concentrated on an orbit (depending on the measure), in which case the analysis of the representations of $C^*(X, G)$ reduces to the analysis of the unitary representations of the isotropy groups along the lines just discussed, or there is an ergodic quasi-invariant measure on X which assigns measure zero to each orbit. In this second case, the measure is called *properly* ergodic. As we will see later, in the presence of free action, properly ergodic measures exist if and only if $C^*(X, G)$ is *not* type I. Also, as we shall see, having properly ergodic measures is a property that can be detected directly in terms of the action of G on X.

Need a reference to Chapter 6 here.

To pursue an analysis of properly ergodic actions, Mackey struck upon a truly wonderful idea: One may profitably think of a transformation group $X \times G$ together with a properly ergodic measure, assuming one exists, as a "virtual subgroup" of G (see [**121, 120, 123**]). One then views cocycles as "homomorphisms" of this virtual subgroup. As a result, J becomes a modular function, Θ becomes a unitary representation, and his Imprimitivity Theorem generalizes as Theorem 1.14 to say that *every representation of G that arises as part of a (properly ergodic) covariant representation of $C^*(X, G)$ is induced from a representation of a virtual subgroup of G.* Moreover, the uniqueness result, Theorem 1.15, effectively translates the correspondence between commutants discussed in Theorem 1.20.

The harmonic analysis and representation theory of virtual groups can be developed in a fashion quite parallel to that of ordinary harmonic analysis. (In this connection, special attention should be drawn to the works of Arlan Ramsay cited in the references.) Moreover, this theory leads directly to the study of groupoids, which we take up in the next chapter. Consequences from the virtual group perspective will arise in these notes quite frequently (we already have indicated some), but we make no effort to pursue it in a systematic fashion. Most of what we have to say about them will be found in Chapter 4. Our emphasis will be on how groupoids generalize the notion of "matrix indices."

We note, too, that Marc Rieffel [**177**] realized that Mackey's Imprimitivity Theorem, Theorem 1.20, is really an instance of, and can be explained by, Morita theory. This will play an influential role in much of the material to follow.

CHAPTER 2

Groupoids and Groupoid C^*-algebras

The first section of this chapter is devoted to presenting the algebraic fundamentals of groupoids. It will develop that in many respects, groupoids are no more complicated than groups. The key idea to be presented here and emphasized throughout the remainder of the monograph is that in the same fashion as one thinks of groups as acting on sets, one should view groupoids as acting on *fibred* sets. In our opinion, the richness of the subject stems from the imposition of topologies and Borel structures on groupoids. These are introduced in the second section. The third section is devoted to introductory facts about groupoid C^*-algebras.

1. The Algebra of Groupoids

We begin with the definition of groupoid. While in one sense the connection with groups is clear, by itself the definition may seem unmotivated and its salient features may be a bit difficult to grasp. It is best to keep in mind a number of examples. Some will be presented immediately following the definition and the introduction of some useful terminology.

DEFINITION 2.1. *Let G be a set and let $G^{(2)}$ be a subset of $G \times G$. Suppose there is a map $(x, y) \to xy$ from $G^{(2)}$ to G and an involution $x \to x^{-1}$ on G such that the following conditions hold:*

(i) *If (x, y) and (y, z) are in $G^{(2)}$, then so are (xy, z) and (x, yz), and the equation, $(xy)z = x(yz)$, is satisfied;*
(ii) *For all $x \in G$, $(x^{-1}, x) \in G^{(2)}$ and if $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$ while $(xy)y^{-1} = x$.*

Then G, with this structure, is called a *groupoid*. The set $G^{(2)}$ is called the set of *composable pairs* and x^{-1} is called the *inverse* of x.

The maps r and s on G, defined by the formulae $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$, are called the *range* and *source* maps. It follows easily from the definition that they have a common image called the *unit space* of G, which is denoted $G^{(0)}$. Condition (ii) implies that $r(x)x = xs(x) = x$. Consequently, $r(x)$ is called the *left unit* of x and $s(x)$ is called the *right unit*. It is useful to note that a pair (x, y) lies in $G^{(2)}$ precisely when $r(y) = s(x)$ and that the elements of the *unit space* $G^{(0)}$ are characterized by the equation $x = x^{-1} = x^2$. Because of this equation, we like to think of units as (infinitesimal) orthogonal projections and we like to think of elements of G as (infinitesimal) partial isometries; the initial space of $x \in G$, viewed this way, is $s(x)$, while the final space is $r(x)$. We will describe many examples throughout this monograph. The following initial collection identifies examples that will occur frequently throughout this monograph.

EXAMPLES 2.2. (i) *Of course, every group is a groupoid. In a group, any two elements are composable and the unit space reduces to a singleton consisting of the identity element.*

(ii) *Suppose H is a group that acts on a set X, as in the first chapter. Let $G = X \times H$, set $G^{(2)} = \{((x,t),(y,s)) \mid y = xt\}$, define $(x,t)(xt,s) = (x,ts)$ and set $(x,t)^{-1} = (xt, t^{-1})$. Then it is routine to check that with these operations G becomes a groupoid. Observe that $r(x,t) = (x,e)$, while $s(x,t) = (xt,e)$, so we may view X as the unit space of G. We shall frequently make this identification. This groupoid is called the* transformation group groupoid *determined by the action of H on X.*

EXAMPLE 2.3. (iii) *Let X be any set and let $R \subseteq X \times X$ be an equivalence relation. Set $R^{(2)} = \{((x_1,y_1),(x_2,y_2)) \mid y_1 = x_2\}$, define $(x_1,y_1)(y_1,y_2) = (x_1,y_2)$ and define $(x,y)^{-1} = (y,x)$. Then it is easy to check that R becomes a groupoid and it is evident that $R^{(0)}$ is the diagonal $\Delta = \{(x,x) \mid x \in X\}$. We frequently identify X with $\Delta = R^{(0)}$ in this context. Two extreme cases deserve to be singled out. If $R = X \times X$, then R is called the* trivial groupoid *on X, while if $R = \Delta$, then R is called the* co-trivial groupoid *on X.*

(iv) *Let X be a set and let $p : X \to U$ be a surjective map from X to another set U. We say that X is fibred by p over U, and we call $p^{-1}(u)$ the fibre over u, writing X_u or $X(u)$ for $p^{-1}(u)$. Let $\mathrm{Iso}(X,p,U)$ be the set $\{(v,\phi,u) \mid \phi : X_u \to X_v$ is a bijection$\}$. We say that two elements (v_1,ϕ_1,u_1) and (v_2,ϕ_2,u_2) are composable in case $u_1 = v_2$. In this event, we define $(v_1,\phi_1,u_1)(u_1,\phi_2,u_2) = (v_1,\phi_1\circ\phi_2,u_2)$. Also, $(v,\phi,u)^{-1}$ is defined to be (u,ϕ^{-1},v). Then with respect to these operations $\mathrm{Iso}(X,p,U)$ is easily seen to be a groupoid, which is called the* isomorphism groupoid *of the fibred set X. As we shall see in a moment, there is a "Cayley Theorem" for groupoids, asserting that every groupoid is isomorphic to a subgroupoid of $\mathrm{Iso}(X,p,U)$ for a suitable fibred set X.*

(v) *Let $\mathcal{G} = \{G_u\}_{u\in U}$ be a family of groups indexed by a set U. We let $U * \mathcal{G} = \{(u,t) \mid u \in U,\ t \in G_u\}$. (The process of forming $U * \mathcal{G}$ is the standard way of forming the "disjoint union" of the family $\mathcal{G}$.) If we define $(U*\mathcal{G})^{(2)}$ to be $\{((u,t),(v,s)) \mid u = v\}$, and if we set $(u,t)(u,s) = (u,ts)$, and $(u,t)^{-1} = (u,t^{-1})$, then $(U*\mathcal{G})$ becomes a groupoid, called a bundle of groups over U. Of course $U*\mathcal{G}$ is fibred over U in the obvious way, $p(u,t) = u$, and $p^{-1}(u)$ is naturally identified with G_u.*

(vi) *Let $\mathcal{Q}$ and $\mathcal{V}$ be sets, and suppose that there are given two surjective maps r and s from $\mathcal{Q}$ to $\mathcal{V}$. The set $\mathcal{Q}$, or the system $(\mathcal{Q},\mathcal{V},r,s)$, is called a* quiver. *The elements of $\mathcal{Q}$ are called* arrows *and the elements of $\mathcal{V}$ are called* vertices. *For $q \in \mathcal{Q}$, $r(q)$ is called the* range *of q and $s(q)$ is called the* source *of q.*[1] *Associated with a quiver is a* path space *X and a* path groupoid *$\mathcal{G}$ defined as follows. The space X consists of all infinite sequences $x = (q_1,q_2,\dots)$ with the property that $s(q_i) = r(q_{i+1})$ for all i. The groupoid $\mathcal{G}$ consists of all triples $(x,l,y) \in X \times \mathbb{Z} \times X$, $x = (q_1,q_2,\dots)$ and $y = (q_1',q_2',\dots)$, with the property that there is an integer N, depending on (x,l,y), such that for all $k > N$, $q_k = q'_{k-l}$. Two triples, (x_1,l_1,y_1) and (x_2,l_2,y_2),*

[1] An alternate term for a quiver is a directed graph; the vertex set is $\mathcal{V}$ and the edge set is $\mathcal{Q}$. However, we have adopted the terminology of quivers out of deference to Gabriel, who first introduced path space techniques into algebra in [**74**]. See [**75**] also.

are declared composable precisely when $y_1 = x_2$ *and their product, then, is* $(x_1, l_1 + l_2, y_2)$. *The inverse of* (x, l, y) *is* $(y, -l, x)$. *As we shall see later, the class of groupoids* $\mathcal{G}$ *associated with finite quivers coincides with those groupoids giving rise to Cuntz-Krieger* C^**-algebras. In the special case when* $\mathcal{V}$ *consists of only one point and* $\mathcal{Q}$ *has cardinality* n *then the groupoid* $\mathcal{G}$*, gives the Cuntz* C^**-algebra* $\mathcal{O}_n$.

Another way to think of a groupoid is to say that a groupoid is a small category in which every morphism is invertible. This perspective is very useful and helps to inform one of the kind of algebraic operations that can be performed with groupoids. It is the point of view adopted in Higgins's book [**92**]. We will not try to make a comprehensive presentation of the basic algebraic facts about groupoids here, but some of aspects will be necessary.

DEFINITION 2.4. (i) *For* $i = 1, 2$, *let* G_i *be a groupoid. A map* $\phi : G_1 \to G_2$ *is a* homomorphism *if and only if whenever* $(x, y) \in G_1^{(2)}$, $(\phi(x), \phi(y))$ *is in* $G_2^{(2)}$, *and in this case,* $\phi(xy) = \phi(x)\phi(y)$.

(ii) *For* $i = 1, 2$, *let* $\phi_i : G_1 \to G_2$ *be homomorphisms. We say that* ϕ_1 *is* similar *or* cohomologous *to* ϕ_2 *if and only if there is a function* $b : G_1^{(0)} \to G_2$ *such that* $b(r(x))\phi_1(x)b(s(x))^{-1} = \phi_2(x)$ *for all* $x \in G_1$.

(iii) *If* $\phi(x) = b(r(x))b(s(x))^{-1}$, *for a function* $b : G_1^{(0)} \to G_2$, *then* ϕ *(or* b*) is called a* coboundary.

It is easy to see that homomorphisms map units to units and inverses to inverses. Also, "similarity" generalizes the notion of conjugacy or similarity for group homomorphisms. That is, if G_1 and G_2 are groups, then two homomorphisms $\phi, \psi : G_1 \to G_2$ are similar in the sense of Definition 2.4 if and only if there is a $b \in G_2$ such that $\psi(x) = b\phi(x)b^{-1}$ for all $x \in G_1$. If $G_1 = X \times H$, as in Example 2.2(ii), and if G_2 is a group, then a homomorphism is simply a map $\phi : X \times H \to G_2$ such that $\phi(x, ts) = \phi((x, t)(xt, s)) = \phi(x, t)\phi(xt, s)$. That is, homomorphisms in this context are (strict) 1-cocycles in the sense of Chapter I. (See equation (1.4) in particular.) Two such homomorphisms ϕ_1 and ϕ_2 are cohomologous or similar if and only if $\phi_2(x, t) = b(x)\phi_1(x, t)b(xt)^{-1}$ (cf. equation (1.5)).

If one takes the perspective that a groupoid is a small category in which every morphism is invertible, then homomorphisms are *functors* and cohomologous homomorphisms are *naturally equivalent* functors, with a coboundary defining a *natural transformation* between them.

The following proposition will be cited at several points in these notes.

PROPOSITION 2.5. *If* $G_1 = X \times X$ *is the trivial groupoid on the set* X, *then any homomorphism,* $\phi : G_1 \to G_2$, *from* G_1 *to any groupoid* G_2 *is a coboundary.*

PROOF. Fix $y_0 \in X$ arbitrarily and define $b : X(= G_1^{(0)}) \to G_2$ by the equation $b(x) = \phi(x, y_0)$. Then for $(x, y) \in G_1$,

$$\begin{aligned}\phi(x,y) &= \phi((x,y_0)(y_0,y)) \\ &= \phi(x,y_0)\phi(y_0,y) \\ &= \phi(x,y_0)\phi((y,y_0)^{-1}) \\ &= \phi(x,y_0)(\phi(y,y_0))^{-1} \\ &= b(x)b(y)^{-1} \\ &= b(r(x,y))b(s(x,y))^{-1}.\end{aligned}$$

□

Let G be a groupoid and define $\phi : G \to G^{(0)} \times G^{(0)}$ by the formula $\phi(x) = (r(x), s(x))$. A moment's reflection reveals that the range of ϕ is an equivalence relation in $G^{(0)} \times G^{(0)}$ and that ϕ is a homomorphism when the equivalence relation is viewed as a groupoid. Observe that if $G = X \times H$ is a transformation group and if X is identified with $G^{(0)}$, then $\phi : G \to X \times X$ is given by the formula $\phi(x, t) = (x, xt)$. Thus the relation $\phi(G)$, in this case, is the so-called *orbit equivalence relation* determined by G. This observation and the terminology from the theory of transformation groups motivate the definitions to follow. The reader should take note that the terminology is not universal. However, in the application of groupoids to operator algebra, the terminology is commonly accepted.

DEFINITION 2.6. *Let G be a groupoid and let $\phi(x) = (r(x), s(x)), x \in G$.*

(i) *The range of ϕ in $G^{(0)} \times G^{(0)}$ is called the* orbit equivalence relation *determined by G.*
(ii) *For $u \in G^{(0)}$, its equivalence class under $\phi(G)$ is called the* orbit *of u.*
(iii) *A subset $A \subseteq G^{(0)}$ is called* invariant *if and only if it is saturated with respect to $\phi(G)$, i.e., if and only if it is a union of orbits.*
(iv) *G is called a* principal *groupoid if and only if ϕ is 1-1. In this case, we often identify G with $\phi(G)$ and also call G an equivalence relation. (Note that a transformation group groupoid $X \times H$ is principal if and only if H is freely acting.)*
(v) *A groupoid is* transitive *if and only if ϕ is onto, i.e., if and only if there is only one orbit.*

The proof of the following proposition is evident and so will be omitted.

PROPOSITION 2.7. *Let $\{G_\alpha\}_{\alpha \in A}$ be a family of groupoids.*

(i) *Assume $\{G_\alpha\}_{\alpha \in A}$ is disjoint and let $G = \bigcup_{\alpha \in A} G_\alpha$. Then G becomes a groupoid if $G^{(2)}$ is defined to be $\bigcup_{\alpha \in A} G_\alpha^{(2)}$ and if the operations are defined in the obvious way.*
(ii) *Let $G = \prod_{\alpha \in A} G_\alpha$. Then G becomes a groupoid if $G^{(2)}$ is defined to be $\{((x_\alpha),(y_\alpha)) \mid (x_\alpha, y_\alpha) \in G_\alpha^{(2)}\}$; the operations on G are defined componentwise.*

In view of this proposition, it is worth noting that when viewed as a groupoid every equivalence relation is a disjoint union of trivial groupoids; i.e., the groupoid is the disjoint union of the trivial groupoids built on the equivalence classes.

In conjunction with the preceding proposition, the following self-evident proposition is one indication of why groupoids are a more flexible tool in operator algebra than transformation groups.

PROPOSITION 2.8. *Let G be a groupoid and let E be a (non-empty) subset of $G^{(0)}$. Then if $G|_E$ is defined to be $r^{-1}(E) \cap s^{-1}(E)$, $G|_E$ becomes a groupoid with unit space E when $(G|_E)^{(2)}$ is defined to be $G^{(2)} \cap (G|_E \times G|_E)$ and the operations are the restrictions of the operations on G.*

One should think of the process of passing from G to $G|_E$ as the process of taking a corner of a matrix. The groupoid $G|_E$ is called the *reduction* or *contraction* of G by E. Note that G is the (disjoint) union of the groupoids $G|_E$ and $G|_{E^c}$ precisely when E is invariant. It is particularly important to note that if E is a singleton $\{u\}$, $u \in G^{(0)}$, then $G|_{\{u\}}$ is a group with identity u. This group is called the *isotropy group* or *stability group* of u. When G is a transformation group groupoid, this new terminology agrees with that in Chapter I.

It is perhaps worthwhile to insert here that while it is tempting to think of $G|_E$ as a *subgroupoid* of G, we reserve the term, instead, for a subset $H \subseteq G$, *containing* $G^{(0)}$, such that when $H^{(2)}$ is defined to be $(H \times H) \cap G^{(2)}$, H is closed under the operations of G. The issues of subobjects and quotient objects in the category of groupoids is rather more complicated than those in the category of groups. We will have little to say about them here and refer the reader to [**29**], [**92**], and [**113**] for further discussion. There is, however, one point concerning these notions that will be of particular use to us.

DEFINITION 2.9. *Let G be a groupoid and let $\Im = \{x \mid r(x) = s(x)\}$. Then $\Im$ is called the* isotropy group bundle *of G.*

Evidently, $\Im$ is the disjoint union $\bigcup_{u \in G^{(0)}} G|_{\{u\}}$, and so is a bundle of groups in the sense of Example 2.2(v). Also, $\Im$ is a subgroupoid of G in the sense just defined. Recalling that the map $\phi : G \to G^{(0)} \times G^{(0)}$ defined by the formula $\phi(x) = (r(x), s(x))$ is a homomorphism and noting that its kernel (i.e., $\phi^{-1}((G^{(0)} \times G^{(0)})^{(0)}) = \phi^{-1}(\triangle)$, where $\triangle$ is the diagonal in $G^{(0)} \times G^{(0)}$), is the isotropy groupoid $\Im$, we are inclined to write

$$G^{(0)} \longrightarrow \Im \longrightarrow G \longrightarrow \phi(G) \longrightarrow G^{(0)} \tag{1.1}$$

to summarize all of this succinctly, in analogy with the custom in group theory. Here, of course, the first two arrows indicate the obvious imbeddings and the third indicates ϕ. However, no sense, really, can be given to the last arrow except to say that it indicates that ϕ is surjective. One adjustment one might make is to replace the right-hand copy of $G^{(0)}$ in the sequence with the quotient space $G^{(0)}/G$ consisting of all the orbits $[u]$ of points $u \in G^{(0)}$. The last arrow then becomes the homomorphism from $\phi(G)$ onto the co-trivial groupoid $G^{(0)}/G$ which sends $(u, v) \in \phi(G)$ to $[u](= [v])$. While this is fine from the algebraic perspective, it meets with difficulties when topological or measure theoretic structures are involved. Consequently, we adopt the *convention* that when writing short exact sequences like (1.1), the last arrow simply indicates that the preceding map is surjective.

It should be noted that the sequence (1.1) leads to a natural gradation in the study of groupoid C^*-algebras. One first concentrates on the principal case; i.e., on the case when all the isotropy groups reduce to singletons and then one worries about how the isotropy groups might be distributed. Caution should be exercised here, however. In many interesting groupoids, one cannot separate the isotropy from the "principal part" in any simple fashion and one just has to take the groupoid as a whole. We will return to this point in due course.

We adopt the following notation in the next proposition and throughout this monograph. If $E \subseteq G^{(0)}$, then $r^{-1}(E)$ is denoted G^E and $s^{-1}(E)$ is denoted G_E. We write G^u and G_u for $G^{\{u\}}$ and $G_{\{u\}}$. This is the standard notation in the literature. However, the following alternate notation, suggested to us by Arlan Ramsay, has its merits: For G^E write $E \cdot G$, where for any pair of subsets $A, B \subseteq G$, $A \cdot B := \{\alpha\beta \mid (\alpha, \beta) \in G^{(2)} \cap (A \times B)\}$. Likewise, G_E may be written as $G \cdot E$ and, in particular, $G^u = uG$ and $G_u = Gu$.

PROPOSITION 2.10. (i) *Each groupoid may be written (uniquely) as the disjoint union of transitive groupoids.*
(ii) *Each transitive groupoid is isomorphic to the Cartesian product of a trivial groupoid and a group.*

PROOF. The first assertion is clear: $G = \bigcup_{[u]\in G^{(0)}/G} G|_{[u]}$ and $G|[u]$ is transitive. (Remember, $G^{(0)}/G$ denotes the quotient space of $G^{(0)}$ determined by the orbit equivalence relation.) For (ii), we show that if G is transitive, then G is isomorphic to $G|_{\{u\}} \times (G^{(0)} \times G^{(0)})$ where $G^{(0)} \times G^{(0)}$ is the trivial groupoid on $G^{(0)}$ and u is *any* unit in $G^{(0)}$. (Recall, $G|_{\{u\}}$ denotes the isotropy group at u.) Fix $u \in G^{(0)}$ and note that since G is transitive, the restriction of s to G^u, $s|G^u$, maps G^u *onto* $G^{(0)}$. Let γ be any cross section to $s|G^u$; i.e., γ maps $G^{(0)}$ to G^u and satisfies $s(\gamma(v)) = v$, $v \in G^{(0)}$. Finally, define $\psi : G \to G|_{\{u\}} \times (G^{(0)} \times G^{(0)})$ by the formula

$$\begin{aligned}\psi(x) &= (\gamma(r(x))x\gamma(s(x))^{-1},\ r(x),\ s(x))\\ &= (\gamma(r(x))x\gamma(s(x))^{-1},\ \phi(x)).\end{aligned}$$

It is easy to check that ψ is well defined and an isomorphism. □

REMARK 2.11. *The decomposition provided by Proposition 2.10 involves making arbitrary choices at two points: First, there is the choice of a representative from each equivalence class in $G^{(0)}/G$; and, second, there is the choice of cross section γ. When there are topologies or measure theoretic structures on a groupoid, it may not be possible to make such choices in a continuous or measurable fashion. These "handicaps" provide a richness in the theory which otherwise would be of limited interest from the perspective of operator algebra.*

PROPOSITION 2.12. *(*Cayley's Theorem for Groupoids*) Every groupoid is isomorphic to a subgroupoid of* Iso(X, p, U) *for a suitable fibred set $p : X \to U$.*

PROOF. Let $X = G$, $p = r$, and $U = G^{(0)}$. Then $X_u = G^u$, $u \in G^{(0)}$. Define $\Phi : G \to \mathrm{Iso}(G, r, G^{(0)})$ by $\Phi(x) = (r(x), \ell_x, s(x))$, where ℓ_x maps $G^{s(x)}$ to $G^{r(x)}$ according to the formula $\ell_x y = xy$. □

One of the fundamental concepts in ring theory, which plays a crucial role also in operator algebra, is *Morita equivalence.* (We will take this up in detail in Chapter 5.) The notion actually appears at the level of groupoids, as we will discuss momentarily. It is related to, and in a purely algebraic sense is equivalent to, the concept of *similar groupoids.* Similar groupoids occur in the measure theoretic setting and will be discussed at greater length in Chapter 4. To expose the algebraic fundamentals of both of these concepts, we begin with the ideas of "fibred product" and "groupoid action."

Recall that if X, Y and Z are sets and if $p_X : X \to Z$ and $p_Y : Y \to Z$ are surjective maps, then the *fibred product* they determine, denoted $X * Y$, is defined to be $\{(x, y) \mid p_X(x) = p_Y(y)\}$. Of course the notation $X * Y$ is a little inadequate. Omitted are any references to Z and to the maps p_X and p_Y. Usually, these are evident from context; the star, $*$, simply alerts the reader that a fibred structure is present. Working with fibred products can sometimes be tricky. It is useful, therefore, to note that "the larger Z is, the smaller $X * Y$ is." At one extreme, when Z reduces to a point, $X * Y$ is $X \times Y$, while if $X = Y = Z$ and $p_X = p_Y$ is the identity, then $X * X$ is the diagonal Δ. Also, it is helpful to keep in mind that given a groupoid G, $G^{(2)}$ is the fibred product $G * G = \{(x, y) \in G \times G \mid s(x) = r(y))\}$.

DEFINITION 2.13. *Let G be a groupoid and let X be a set. We say that G* acts *on X (to the left), and that X is a* left G-space, *in case there is a surjection $r : X \to G^{(0)}$ and a map $(\gamma, x) \to \gamma x$ from $G * X := \{(\gamma, x) \mid s(\gamma) = r(x)\}$ to X such that*

(i) $r(\gamma x) = r(\gamma)$, $(\gamma, x) \in G * X$;
(ii) *if* $(\gamma_1, x) \in G * X$ *and* $(\gamma_2, \gamma_1) \in G^{(2)}$, *then* $(\gamma_2\gamma_1, x)$, $(\gamma_2, \gamma_1 x) \in G * X$ *and* $\gamma_2(\gamma_1 x) = (\gamma_2\gamma_1)x$; *and*
(iii) $r(x)x = x$, $x \in X$.

Right actions and right G-spaces are defined similarly, but we use s to denote the map from X to $G^{(0)}$ and we write $X * G = \{(x, \gamma) \mid s(x) = r(\gamma)\}$.

Of course, to say that G acts on X (on the left) is to say that there is a homomorphism of G into the isomorphism groupoid, $\mathrm{Iso}(X, r, G^{(0)})$, of the set X fibred by r over $G^{(0)}$.

The notation for the maps r and s is to remind one of the range and source maps in a groupoid. The fact that the same notation is used in both situations should not cause confusion. It will be possible to distinguish among the multiple uses of r and s from context.

REMARKS 2.14. 1. *Suppose that the groupoid G acts on the left of X (resp. on the right of X). Then $G * X$ (resp. $X * G$) has the structure of a groupoid called the left* action groupoid *(resp. right action groupoid) determined by the action. The space of composable pairs, $(G*X)^{(2)}$, (resp. $(X*G)^{(2)}$) is defined to be $\{((\gamma_1, x_1), (\gamma_2, x_2)) \mid x_1 = \gamma_2 x_2\}$ (resp. $\{((x_1, \gamma_1), (x_2, \gamma_2)) | x_2 = x_1\gamma_1\}$), with $(\gamma_1, \gamma_2 x_2)(\gamma_2, x_2) = (\gamma_1\gamma_2, x_2)$ and $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma x)$ (resp. $(x_1, \gamma_1)(x_1\gamma_1, \gamma_2) = (x_1, \gamma_1\gamma_2)$ and $(x, \gamma)^{-1} = (x\gamma, \gamma^{-1})$). The unit space of $G * X$ (resp. $X * G$) is identified with X through the map $x \leftrightarrow (r(x), x)$ (resp. $x \leftrightarrow (x, s(x))$). We write $G\backslash X$ (resp. X/G)[2] for the quotient space of X under the relation $x \sim y$ if and only if there is a γ such that $\gamma x = y$ (resp. $x\gamma = y$). Thus action groupoids are obvious generalizations of transformation groups.*

2. *A particular type of action groupoid deserves to be singled out, namely the situation where G acts to the right on itself. This groupoid plays an important role in the theory, as we shall see. In this case, $G * G$ is $G^{(2)}$. We have $(G^{(2)})^{(2)} = \{((x_1, y_1), (x_2, y_2)) | x_2 = x_1 y_1\}$, $(x_1, y_1)(x_1 y_1, y_2) = (x_1, y_1 y_2)$, and $(x, y)^{-1} = (xy, y^{-1})$. Let $r^{(2)}$ and $s^{(2)}$ denote the range and source*

[2] It may be helpful to note that X/G is read 'X over G' while $G\backslash X$ is read as '*G under X*'.

maps on $G^{(2)}$. *Then* $r^{(2)}(x,y) = (x, s(x))$ *and* $s^{(2)}(x,y) = (xy, s(xy))$. *It is evident from this that* $(G^{(2)})^{(0)}$ *may be identified with* G. *If* $r^{(2)}(x,y) = s^{(2)}(x,y)$, *then* $xy = x$, *forcing* $y = s(x)$. *Consequently, in this case,* (x,y) *is the unit* $(x, s(x))$, *showing that* $G^{(2)}$ *is principal. Further, units* $(x, s(x))$ *and* $(y, s(y))$ *lie in the same* $G^{(2)}$*-orbit if and only if* $r(x) = r(y)$. *Thus, we may view* $G^{(2)}$ *as the equivalence relation on* G, *where* $x, y \in G$ *are equivalent iff* $r(x) = r(y)$. *The equivalence classes are the spaces* G^u, *as* u *runs over* $G^{(0)}$, *and* $G^{(0)}$ *is a transversal for the equivalence relation.*

If X is a left G-space, we let X^{op} denote the space X, but with the *right* action of G defined as follows: $s^{\mathrm{op}} = r$ and $x\gamma = \gamma^{-1}x$. Notice that the use of γ^{-1} is necessary; γx does not make sense.

Let X be a left G-space and give $X^{\mathrm{op}} * X = \{(x,y) \mid s^{\mathrm{op}}(x) = r(y)\}$ $(= \{(x,y) \mid r(x) = r(y)\})$ the diagonal action: $\gamma(x,y) = (x\gamma^{-1}, \gamma y)$ $(= (\gamma x, \gamma y))$ (provided $s(\gamma) = s^{\mathrm{op}}(x) = r(y)$). We denote the quotient space, $G\backslash(X * X)$, by $\underline{G}$ or by $X^{\mathrm{op}} *_G X$, and we denote its elements by $[x,y]$, $(x,y) \in X^{\mathrm{op}} * X$. Thus, and this is the key property, $[x\gamma, y] = [x, \gamma y]$. Then $\underline{G}$ becomes a groupoid when $\underline{G}^{(2)}$ is defined to be $\{([x_1,y_1],[x_2,y_2]) |$ there exists $\gamma \in G$ such that $x_2\gamma = y_1\}$. The multiplication is then defined by the formula $[x_1,y_1][x_2,y_2] = [x_1,y_1][y_1\gamma^{-1}, y_2] = [x_1,y_1][y_1,\gamma^{-1}y_2] = [x_1,\gamma^{-1}y_2] = [x_1\gamma^{-1}, y_2]$ and inversion is defined by the formula $[x,y]^{-1} := [y,x]$. (One must check that the multiplication is well defined, but that is not difficult.) Note that the unit space of $\underline{G}$ may be identified naturally with $G\backslash X$, whose elements are denoted $[x]$, $x \in X$, where the range and source maps $\underline{r}$ and $\underline{s}$ are defined by the formulae: $\underline{r}([x,y]) = [x]$, and $\underline{s}([x,y]) = [y]$.

It is helpful to think of $X^{\mathrm{op}} * X$ as an equivalence relation and $\underline{G} = X^{\mathrm{op}} *_G X = G\backslash\,(X^{\mathrm{op}} * X)$ as a kind of quotient. It is also instructive to think of the pair (G,X) as an analogue of (R,M) where R is a ring and M is a left R-module. In this case, $\underline{G}$ becomes the analogue of $M^* \otimes_R M$, where $M^* = \mathrm{Hom}_R(M,R)$. This analogy is further reinforced when one recognizes that $M^* \otimes_R M$ has a ring structure and that M becomes a right $M^* \otimes_R M$ module in a fashion that is perfectly paralleled in the groupoid setting: Define $s : X \to G\backslash X \approx \underline{G}^{(0)}$ to be the quotient map; set $X * \underline{G} := \{(z,[x,y]) \mid s(z) = [x]\} = \{(z,[x,y]) |$ there exists $\gamma \in G$ such that $z = \gamma x$ $(= x\gamma^{-1})\}$; and set $z \cdot [z\gamma, y] = z \cdot [z, \gamma y] := \gamma y$. Then it is easy to check that this action is well defined and X becomes a right $\underline{G}$-space satisfying $\gamma \cdot (z \cdot [x,y]) = (\gamma \cdot z) \cdot [x,y]$; i.e., the actions of G and $\underline{G}$ commute.

There is, of course, a complete left-right duality here. Given a right G-space X, we can turn X into a left G-space, also denoted X^{op}, via the formula: $\gamma x := x\gamma^{-1}$. The map r^{op} is s. One may form the groupoid $\underline{G} = X *_G X^{\mathrm{op}}$ which is the quotient of $X * X^{\mathrm{op}}$ by the diagonal action of G and observe that $X *_G X^{\mathrm{op}}$ acts to the left on X. Again, this action commutes with the original G action.

DEFINITION 2.15. *The groupoid* $\underline{G} = X^{\mathrm{op}} *_G X$ *(resp.* $X *_G X^{\mathrm{op}}$*) associated to a left (resp. right)* G*-space* X *is called the* imprimitivity groupoid *of the pair* (X,G) *or simply of* X *if the role of* G *is clear.*

The generalization of the notion of a free group-action to the groupoid setting is clear: A groupoid G acts *freely* on X (on the left) precisely when the equation $\gamma \cdot x = x$ implies that γ is the unit $s(\gamma) = r(x)$. Similarly, one defines the notion of free right action. Observe that if G acts freely on X, then so does $\underline{G}$, and the pair, $(G, \underline{G})$, are equivalent in the following sense.

DEFINITION 2.16. *Let G and H be groupoids. We say that G and H are* equivalent *in case there is a set X endowed with a free left action of G and a free right action of H such that the actions commute and such that the map r induces a bijection between X/H and $G^{(0)}$ while the map s induces a bijection between $G\backslash X$ and $H^{(0)}$, i.e., $r(x_1) = r(x_2)$ iff there is an $\eta \in H$ such that $x_1\eta = x_2$ and $s(x_1) = s(x_2)$ iff there is $\gamma \in G$ such that $\gamma \cdot x_1 = x_2$.*

We then refer to X an *equivalence* between G and H or simply a (G, H)–*equivalence.*

Strictly speaking we should probably append a parenthetic adjective "algebraic" and adverb "algebraically" to the terms "equivalence" and "equivalent" (and later to "similarity" and "similar"). In the topological (and measure theoretic) settings in which these terms will most frequently be used, additional hypotheses are added. Topological equivalence will be developed in Chapter 5 while measure theoretic similarity will be discussed in Chapter 4.

EXAMPLE 2.17. *An elementary, but nevertheless useful example to keep in mind, is the one where G is the trivial groupoid on $\{1, 2, \ldots, n\}$, $G = \{1, 2, \ldots, n\}^2$, H is $\{1, 2, \ldots, m\}^2$ and X is, then, $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$. The map r is defined to be the left projection, $r(k, l) = k$ while s is the right projection, $s(k, l) = l$. If $(i, j) \in G$, and $(j, k) \in X$, then $(i, j) \cdot (j, k)$ is defined to be (i, k) and a similar definition is given when $(i, j) \in X$ and $(j, k) \in H$. One verifies easily that X is a (G, H)-equivalence.*

This example "contains" the well-known fact that $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$ are Morita equivalent rings.

EXAMPLE 2.18. *Another example to keep in mind relates Morita equivalence to the notion of similarity that we shall discuss in a minute. Let G be a groupoid and let T be a subset of $G^{(0)}$ whose saturation $[T] = s(r^{-1}(T)) = r(s^{-1}(T))$ is all of $G^{(0)}$. Such a set T is sometimes called a* transversal, *although we shall reserve this term for sets T satisfying additional properties. Let $X = G^T$. Since s carries X onto $G^{(0)}$, G acts to the right on X in the obvious fashion: through right multiplication. Likewise, since r maps X to T, the unit space of $G|_T$, it is easy to see that $G|_T$ acts on X to the left via left multiplication, and that X is a $(G|_T, G)$-equivalence.*

REMARKS 2.19.
1. *If G and H are equivalent via X, then H is isomorphic to $\underline{G}$. Indeed, given $[x, y] \in \underline{G}$, we know that $r(x) = s^{\mathrm{op}}(x) = r(y)$, by definition. Hence there is an $\eta \in H$ so that $y\eta = x$. This η is unique, since the action of H is assumed to be free, and the map $[x, y] \to \eta$ is easily seen to be an isomorphism.*
2. *Given a groupoid G and a free left G-space X, there is an easily proved "double commutant" theorem: G is isomorphic to $\underline{\underline{G}}$, where $\underline{\underline{G}}$ has the obvious meaning.*
3. *Equivalence of groupoids is an equivalence relation. Indeed, if X is a (G, H)-equivalence and Y is an (H, K)-equivalence, then $X *_H Y$ is a (G, K)-equivalence, where $X *_H Y$ is the quotient of $X * Y = \{(x, y) \mid s(x) = r(y)\}$ by the diagonal action of $H : ((x, y), \eta) \to (x\eta, \eta y)$.*

The first assertion of the next proposition will be generalized to the topological setting in Chapter 5. The second assertion is considerably more problematic in the topological context.

PROPOSITION 2.20. (i) *If G is a transitive groupoid, then for any unit $u \in G^{(0)}$, G and $G|_{\{u\}}$ are equivalent;*
(ii) *Every groupoid is equivalent to a bundle of groups.*

PROOF. The second assertion follows from the first, since equivalence clearly respects disjoint unions. The first assertion is a special case of Example 2.18. One takes for T in that example, the singleton $\{u\}$. The assumption that the groupoid is transitive is precisely the assumption that $[u] = G^{(0)}$. □

DEFINITION 2.21. *Two groupoids G_1 and G_2 are* similar *in case there are homomorphisms $\phi : G_1 \to G_2$ and $\psi : G_2 \to G_1$ such that $\psi \circ \phi$ is similar to the identity on G_1 and $\phi \circ \psi$ is similar to the identity on G_2; i.e., $\phi \circ \psi(x) = b(r(x))xb(s(x))^{-1}$ for a suitable function $b : G_2^{(0)} \to G_2$ and likewise for $\psi \circ \phi$.*

If groupoids are viewed as categories and homomorphisms are regarded as functors, then "similarity" is simply a special case of the notion of equivalence of categories.

EXAMPLE 2.22. *Continuing with the notation and assumptions of Example 2.18, we show that G and $G|_T$ are similar. For this, choose a cross section f for the restriction of s to G^T, i.e., choose $f : G^{(0)} \to G^T$ to satisfy $s \circ f(u) = u$, $u \in G^{(0)}$. Such a choice may always be made by the axiom of choice. Also, adjust f, if necessary, so that $f(u) = u$ for all $u \in T$. Then the adjusted f is still a cross section to the restriction of s to G^T. Define $\phi : G \to G|_T$ by $\phi(x) = f(r(x))xf(s(x))^{-1}$. Then a moment's reflection shows that ϕ is a homomorphism from G to $G|_T$. Such a homomorphism is called a* reduction *(see Definition 4.13). Then take $\psi : G|_T \to G$ to be the identity map. Since $f(u) = u$, for $u \in T$, $\phi \circ \psi$ is the identity on $G|_T$. On the other hand, $\psi \circ \phi = \phi$ is similar to the identity on G, via f. Thus G and $G|_T$ are similar.*

REMARK 2.23. *It may be helpful to note that the proof of Proposition 1.18 really is a special case of this last example. The map γ produced there is the map f in the example.*

PROPOSITION 2.24. *In the purely algebraic setting, two groupoids are equivalent if and only if they are similar.*

PROOF. Since both notions respect the formation of disjoint union, we may assume our groupoids are transitive. Also, note that two *groups* are similar or equivalent if and only if they are isomorphic. It suffices, then, to note that a transitive groupoid G is similar to $G|_{\{u\}}$ for any unit $u \in G^{(0)}$. But this is just a special case of Example 2.22, with $T = \{u\}$. □

The use of cross sections in the analysis of "similarity" indicates why the concept of "equivalence" is preferable to "similarity" in the topological setting.

2. Topological Groupoids

In this section we develop some of the rudimentary facts about topological groupoids.

DEFINITION 2.25. *Suppose G is a groupoid with a topology and give $G^{(2)}$ the relative product topology coming from $G \times G$. Then G is called a* topological groupoid *in case the map $(x, y) \to xy$ from $G^{(2)}$ to G, and the map $x \to x^{-1}$ on G are* *continuous. If a groupoid G has a Borel structure such that $G^{(2)}$ is a Borel subset of $G \times G$ and the above maps are Borel, then we call G a* Borel groupoid. *If G is a topological groupoid, we view G as having the Borel structure inherited from the topological structure.*

We assume, unless otherwise stated or implied by context, that our topological groupoids are locally compact, Hausdorff, and 2nd countable.

It should be noted that in the important application of groupoids to geometry and, in particular, to situations where "germ groupoids" arise, non-Hausdorff groupoids appear quite naturally. They are, however, locally Hausdorff, meaning that each point has a Hausdorff neighborhood, and they are locally compact. It turns out that this extended level of generality is sufficient to develop most of the theory we describe here. Since extending the general discussion to non-Hausdorff groupoids adds little to the points we are trying to make, we proceed with the Hausdorff hypothesis. However, we will say a few words about non-Hausdorff groupoids later

Do something with non-Hausdorff groupoids and cross reference.

Observe that if G is a topological groupoid, then the range and source maps are continuous, since $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$. It follows that if, in addition, G is Hausdorff, then $G^{(2)} = \{(\alpha,\beta) | s(\alpha) = r(\beta)\}$ is closed in G. Further, the unit space $G^{(0)}$ is a closed subspace of G since it is $\{\gamma \in G | \gamma = r(\gamma)\}$.

EXAMPLES 2.26. 1. *Any locally compact group is a locally compact groupoid, of course.*
2. *Of course, also, a locally compact Hausdorff space, viewed as a cotrivial groupoid is a topological groupoid.*

EXAMPLE 2.27. 1. *If a locally compact group G acts on a locally compact space X, then the transformation group groupoid, $X \times G$, with the product topology, is a locally compact groupoid.*
2. *If X is a locally compact space and if $R \subseteq X \times X$ is an equivalence relation that is locally compact with respect to the relative topology on $X \times X$, then R is a locally compact groupoid. In particular, the trivial groupoid on a locally compact space is a locally compact groupoid. Significantly, however, one often wants to consider relations that are endowed with topologies that are different from the relative topology. These occur in many contexts; frequently, they occur in the context of group (and groupoid) actions. Consider, for example, the action of the integers on the circle described in Example 1.1. If the angle α of rotation is irrational, then the map from $\mathbb{T} \times \mathbb{Z}$ to $\mathbb{T} \times \mathbb{T}$ that sends (z,n) to $(z, e^{i\alpha\pi n}z)$ is an isomorphism between the transformation group groupoid, $\mathbb{T} \times \mathbb{Z}$, and its orbit equivalence relation R in $\mathbb{T} \times \mathbb{T}$. This relation is dense in $\mathbb{T} \times \mathbb{T}$ with respect to the usual product topology on $\mathbb{T} \times \mathbb{T}$, but if one transports the topology from $\mathbb{T} \times \mathbb{Z}$ to R then R becomes a locally compact groupoid.*
3. *Let X be a locally compact space and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathbb{A}}$ be an open cover of X. Let $X^{\mathcal{U}}$ be the disjoint union of the sets in $\mathcal{U}$; i.e., $X^{\mathcal{U}} = \{(x,\alpha) | x \in U_\alpha\}$ and let $G^{\mathcal{U}} = \{((x,\alpha),(x,\beta)) \, | x \in U_\alpha \cap U_\beta\}$. Then, of course, $X^{\mathcal{U}}$ is a locally compact space and, with the relative topology, $G^{\mathcal{U}}$ is locally compact. It is a locally compact groupoid with respect to the operations defined as follows: $\left(G^{\mathcal{U}}\right)^{(2)} = \{(((x,\alpha),(x,\beta)),((y,\gamma),(y,\delta)))\,| x = y,\ \beta = \gamma\}$; the product is given by the equation $((x,\alpha),(x,\beta)) \cdot ((x,\beta),(x,\delta)) = ((x,\alpha),(x,\delta))$; and*

$((x,\alpha),(x,\beta))^{-1} = ((x,\beta),(x,\alpha))$. Of course, this groupoid is a special case of the preceding example. However, it will turn out to be very useful later on. For the record, we note that the groupoid cohomology of $G^{\mathcal{U}}$ (to be discussed later) is the same as the topological (Čech) cohomology of $\mathcal{U}$. (One has to specify the coefficients properly.) This observation was made by Mackey in [**120**] *and developed somewhat by Westman in* [**202**]. *See also* [**104**].

4. *The previous example is also a special case of the situation we describe now: Let G be a locally compact groupoid, let X be a locally compact space and suppose there is given a continuous map $\psi : X \to G^{(0)}$. Let $G^\psi = \{(x,\gamma,y) | r(\gamma) = \psi(x),\ s(\gamma) = \psi(y)\}$. With the relative topology from $X \times G \times X$, G^ψ is a locally compact space and it is a locally compact groupoid when the operations are defined by the formulae: $(x,\alpha,y)(y,\beta,z) = (x,\alpha\beta,z)$ (no other pairs are composable), and $(x,\gamma,y)^{-1} = (y,\gamma^{-1},x)$.*

To develop an algebraic theory of functions on groupoids, one needs to hypothesize the existence of an analogue of Haar measure, that we now define.

DEFINITION 2.28. *A* (left) Haar system *on a groupoid G is a family $\{\lambda^u\}_{u\in G^{(0)}}$ of non-negative (Radon) measures on G such that*

(i) $\mathrm{supp}(\lambda^u) = G^u,\ u \in G^{(0)}$;
(ii) *for $f \in C_c(G)$, the function*

$$u \to \int f d\lambda^u$$

on $G^{(0)}$ is in $C_c(G^{(0)})$; and

(iii) *for $x \in G$, $x\lambda^{s(x)} = \lambda^{r(x)}$, i.e.; $\int f(xy)\, d\lambda^{s(x)}(y) = \int f(y)\, d\lambda^{r(x)}(y)$.*

We shall see plenty of examples of Haar systems shortly and throughout the course of the rest of the monograph, but at the outset, it may be helpful to reflect a little on the meaning of the conditions in the definition. First, observe that condition (ii) implies that the map $\Lambda : C_c(G) \mapsto C_c(G^{(0)})$ defined by the formula $\Lambda(f)(u) = \int f d\lambda^u$ is a continuous linear map from $C_c(G)$ to $C_c(G^{(0)})$ where each space is given the inductive limit topology. Further, it is a module map over $C_c(G^{(0)})$ in the sense that $\Lambda(\varphi \cdot f)(u) = \varphi(u)\Lambda(f)(u)$ for all $f \in C_c(G)$ and all $\varphi \in C_c(G^{(0)})$, where $\varphi \cdot f(x) = \varphi(r(x))f(x)$. Conversely, each such module map Λ is determined by a family of measures $\{\lambda^u\}_{u\in G^{(0)}}$ where $\mathrm{supp}(\lambda^u) \subseteq G^u,\ u \in G^{(0)}$ and the continuity condition (ii) is satisfied. Assuming condition (i) is tantamount to assuming that Λ is surjective. Condition (iii) means that Λ is equivariant in an obvious sense. We shall have more to say about Haar systems and families of measures related to them later.

This will have to be in Chapter V or Chapter VII - probably Chapter VII

Unlike the case for groups, Haar systems need not exist, as we shall see in a minute. Also, when a Haar system does exist, it need not be unique in any obvious sense. To illustrate this second point, in a fairly emphatic fashion, consider

What to do about Connes's theorem relating measurable Haarsystems?

EXERCISE 2.29. *Let G be the trivial groupoid on a locally compact Hausdorff space X; i.e., let $G = X \times X$ with the product topology. Then if λ is a fixed measure on X with full support, i.e., $\mathrm{supp}(\lambda) = X$, and if λ^x is defined to be $\epsilon_x \times \lambda$, then $\{\lambda^x\}_{x\in X}$ is a left Haar system on G. Conversely, every Haar system $\{\lambda^x\}_{x\in X}$ on G may be written in this form for a positive measure λ with full support.*

While it is very often the case that Haar systems are not unique, when they exist, in most cases of interest, there is a natural choice of Haar system. This is the case for transformation groups (see below) and groupoids of geometric interest such as foliations and the tangent groupoid of a manifold. We will have more to say about these later. IOU

REMARK 2.30. *The existence of a Haar system implies somewhat more regularity in a locally compact groupoid than is carried by the definition alone: If a locally compact groupoid has a Haar system, then the range and the source maps are open. This fact was asserted by Westman in* [**201**] *and by Renault in* [**171**]. *A proof may be found in* [**195**]. *A minor modification of it works even in the non-Hausdorff setting* [**145**, Proposition 2.2.1].

The following proposition is Lemma 1.3 of [**175**]. It gives a conditioned converse to this remark. First note that if G is a bundle of groups (see Example 2.2) and a locally compact groupoid, then each of the groups $G|_u$ is a locally compact group, and so carries an essentially unique Haar measure, μ_u. Of course the μ_u are not really unique; any two are scalar multiples of one another. It is natural to hope, however, that choices could be made so that they all could be pasted together to yield a Haar system on G. The next proposition determines exactly when this can happen.

PROPOSITION 2.31. *If G is a locally compact groupoid that is a bundle of groups, then G admits a Haar system if and only if the map $p : G \to G^{(0)}$ is open. In that event, a Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ has the property that λ^u is a Haar measure on $G|_u$.*

In [**195**], Seda shows that the following bundle of groups $\mathcal{G}$ does not admit a Haar system: $\mathcal{G} = \{(t,z) \in [0,1] \times \mathbb{T} | z = 1, \text{ when } t > 1/2\}$. Of course, in this example, the bundle map p is not open.

One might speculate that having open range and source maps is a sufficient condition for a groupoid to have a Haar system. However, this is not the case. It sometimes happens that if G is a locally compact groupoid and if R is its orbit equivalence relation, i.e., the image of G in $G^{(0)} \times G^{(0)}$ under the map $\gamma \to (r(\gamma), s(\gamma))$, then G can have a Haar system, while R does not. The groupoid R, however, does have open range and source maps. Work out the details for the Cuntz groupoid or $\mathbb{T}$ acting on the plane $\mathbb{C}$ by rotations.

REMARK 2.32. *Groupoid actions on locally compact spaces and on other topological spaces will play especially important roles in these notes. The notion of groupoid action is defined in Definition 2.13; we keep the notation there. To say that a left action is continuous, it is required that the map $r : X \to G^{(0)}$ be continuous and* open *and that the map $(\gamma, x) \to \gamma x$ from $G * X$ to X be continuous. Likewise, in the case when G acts on the right of X, s must be continuous and open and the map $(x,\gamma) \to x\gamma$ from $X * G$ to X must be continuous.. If X and G are locally compact, then it is easy to see that so are the action groupoids $G * X$ and $X * G$. If, further, $\{\lambda^u\}_{u\in G^{(0)}}$ is a Haar system for G, then $\{\lambda^{r(x)} \times \varepsilon_x\}_{x\in X}$ is a Haar system for $G * X$,while for $X * G$, $\{\varepsilon_x \times \lambda^{s(x)}\}_{x\in X}$ is a Haar system. In particular, note that if G is a group with Haar measure λ, then $\{\varepsilon_x \times \lambda\}_{x\in X}$ is a Haar system for the transformation group groupoid $X \times G$. Also note in particular that the groupoid $G^{(2)}$, which arises when G acts on G to the right (see Remark 2.14) has a Haar system $\{(\lambda^2)^x\}_{x\in G}$, with $(\lambda^2)^x := \varepsilon_x \times \lambda^{s(x)}$.*

There is a notion of imprimitivity groupoid in the topological setting and it has a Haar system under suitable hypotheses. We will take these matters up in detail in Chapters 5 and 7.

A very important class of groupoids are the so-called *r-discrete* groupoids. These are groupoids G such that $G^{(0)}$ is open in G. (We have noted above that $G^{(0)}$ is always closed.) Such groupoids are generalizations of transformation groups $X \times H$, where H is assumed to be discrete. The presence of a Haar system on an r-discrete groupoid forces a bit more structure on the groupoid. We state, for future reference, the following proposition that may be found in [**171**], I.2.7 and I.2.8. It explains the terminology and identifies when a Haar system exists.

PROPOSITION 2.33. *Suppose G is an r-discrete groupoid.*

(i) *For each $u \in G^{(0)}$, G_u and G^u, with the relative topologies, are discrete spaces.*
(ii) *If $\{\lambda^u\}_{u\in G^{(0)}}$ is a Haar system on G, then each λ^u is a multiple of counting measure on G^u.*
(iii) *A Haar system exists on G if and only if r and s are local homeomorphisms.*

Thus, an r-discrete principal groupoid admitting a Haar system is, effectively, an equivalence relation R contained in $X \times X$, where X is locally compact and Hausdorff, such that R has countable equivalence classes and is endowed with a topology (**possibly different from the relative topology**) such that the left and right projections of R onto X are local homeomorphisms.

It is also worthwhile to note that nowadays, the term 'r-discrete groupoid' is taken to mean a locally compact groupoid on which the range and source maps are local homeomorphisms. In the literature, r-discrete groupoids are also called *étale groupoids* [**?**] and *sheaf groupoids* [**103**].

We conclude this section with several results about the structure of the neighborhoods of $G^{(0)}$, in a locally compact groupoid G. These are reminiscent of familiar facts from analysis on locally compact groups, but it takes a bit more work to establish them. First a definition.

DEFINITION 2.34. *A subset L of a topological groupoid G is called* r-relatively compact *in case $L \cap r^{-1}(K)$ is relatively compact (i.e., has compact closure in G) for each compact subset $K \subseteq G^{(0)}$.* s-relatively compact *sets are defined similarly.*

It is helpful to keep in mind the example where G is the open unit square and L is an open strip with sides parallel to the diagonal.

REMARK 2.35. *Observe that if L is r-relatively compact, and if K is relatively compact in G, then $K \cdot L = K \cdot (L \cap r^{-1}(s(K)))$ is relatively compact in G. Also, if G acts continuously (to the left) on a space X and if L is s-relatively compact subset of G, then for all relatively compact subsets $K \subseteq X$, $L \cdot K = (L \cap s^{-1}(r(K))) \cdot K$ is relatively compact in X.*

LEMMA 2.36. [**171**, Proposition 2.1.9] *Suppose G is a second countable, locally compact, Hausdorff groupoid. Then $G^{(0)}$ has a fundamental system of s-relatively compact neighborhoods.*

PROOF. Let V be an open neighborhood of $G^{(0)}$ and let $\{K_i\}$ be a locally finite cover of $G^{(0)}$ consisting of sets that are open in $G^{(0)}$ and have compact closures. Such a choice is possible since the hypotheses on G guarantee that $G^{(0)}$ is paracompact. Choose open sets U_i in G such that each U_i contains K_i, $\overline{U}_i$ is compact in G, and such that $K_i \subseteq \overline{U}_i \subseteq V \cap s^{-1}(K_i)$. Then $U := \cup U_i$ is an open neighborhood of $G^{(0)}$, contained in V, and U is s-relatively compact. This is because any compact set $K \subseteq G^{(0)}$ meets only a finite number of the K_i's, and so $U \cap s^{-1}(K)$ is contained in the union of finitely many U_i, which is relatively compact in G. □

DEFINITION 2.37. *A neighborhood W of $G^{(0)}$ is called is called* diagonally compact *(resp.* conditionally compact*) if VW and WV are compact (resp. relatively compact) for every compact (resp. relatively compact) set V in G. As with groups, W is called* symmetric *in case $W = W^{-1}$.*

LEMMA 2.38. [**138**, Lemma 2.7] *If G is a second countable, locally compact, Hausdorff groupoid, then G has a fundamental system of symmetric, open, conditionally compact neighborhoods. In fact, if W_1 is any neighborhood of $G^{(0)}$, then there is an open, symmetric, conditionally compact set W_0, with $\overline{W}_0$ diagonally compact, such that*

$$G^{(0)} \subseteq W_0 \subseteq \overline{W}_0 \subseteq W_1.$$

PROOF. By Lemma 2.36, $G^{(0)}$ has a fundamental system of open, s-relatively compact neighborhoods. If U is such a neighborhood, then U^{-1} is open and r-relatively compact. So $W_0 := U \cap U^{-1}$ is symmetric and conditionally compact by Remark 2.35. This observation shows, too, that $\overline{W}_0$ is diagonally compact. The fact that we can arrange things so that $\overline{W}_0 \subseteq W_1$ follows from the fact that G is a normal topological space — a consequence of our second countability assumption, together with the assumption that G is locally compact. □

The next lemma was shown to us by Dana Williams.

LEMMA 2.39. *With our standing hypotheses on G, let U be an open neighborhood of $G^{(0)}$, and suppose that W is a diagonally compact neighborhood of $G^{(0)}$ contained in U. Then for every $u \in G^{(0)}$, there is a neighborhood V_u of u in G so that $V_u W \subseteq U$.*

PROOF. Fix $u \in G^{(0)}$, suppose that no such neighborhood V_u may be found, and let A be any compact neighborhood of u. Then for each open neighborhood V of u, we can find a $\gamma_V \in V$ and an $\eta_V \in W$ so that $\gamma_V \eta_V \notin U$. By construction, the net $\{\gamma_V\}$ converges to u. On the other hand, the net $\{\gamma_V \eta_V\}$ is eventually in AW, a compact subset of G by hypothesis. Passing to a subnet and relabeling, if necessary, we may assume that $\{\gamma_V \eta_V\}$ converges to η, say, in AG. But then, necessarily, $\{\eta_V\}$ converges to η. Thus $\eta \in W$, since W is compact. Since $\gamma_V \eta_V \to \eta \in W \subseteq U$, $\{\gamma_V \eta_V\}$ is eventually in U, contrary to our initial assumption. □

The following proposition is a natural extension to groupoids of a well-known fact from the theory of topological groups. In contrast to the group setting where the proof is a simple consequence of the continuity of multiplication, the proof for groupoids seems to require all that we have done in the last three lemmas. The proof is due to Dana Williams.

PROPOSITION 2.40. *If G is a second countable, locally compact, Hausdorff groupoid, and if U is a neighborhood of $G^{(0)}$ in G, then there is a neighborhood V of $G^{(0)}$ in G such that $V^2 \subseteq U$.*

PROOF. First use Lemma 2.38 to choose a diagonally compact neighborhood W of $G^{(0)}$ contained in U. Then use Lemma 2.39 to find a (countable) cover $\{V_n\}$ of $G^{(0)}$ by open sets in G so that $V_n W \subseteq U$, and $V_n \subseteq W$. Then $V := \cup_{n=1}^{\infty} V_n \subseteq W$, and $V^2 \subseteq VW = \cup_{n=1}^{\infty} V_n W \subseteq U$. □

3. Groupoid algebras

Once a Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ has been specified on a groupoid, we may define an involutive algebraic structure on $C_c(G)$ by the formulae:

$$f * g(y) = \int f(x)g(x^{-1}y)\, d\lambda^{r(y)}(x)$$
$$= \int f(yx)g(x^{-1})\, d\lambda^{s(y)}(x),$$

and

$$f^*(x) = \overline{f(x^{-1})},$$

$f, g \in C_c(G)$. It is easy to verify that with respect to these operations and the inductive limit topology $C_c(G)$ is a topological $*$-algebra. (If we wish to emphasize the role of λ in this structure, we write $C_c(G, \lambda)$). Examples will be given in a moment, but first we want to introduce the primary objects of study in these notes.

DEFINITION 2.41. *A* representation *of $C_c(G)$ is a *-homomorphism π from $C_c(G)$ into $B(H)$, for some Hilbert space H, that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $B(H)$, and that is nondegenerate in the sense that the closed linear span of $\{\pi(f)\xi \mid f \in C_c(G), \xi \in H\}$ is all of H.*

THEOREM 2.42. *For $f \in C_c(G)$, the quantity*

$$\|f\| := \sup\{\, \|\pi(f)\| \mid \pi \text{ — a representation}\}$$

is finite and defines a C^*-norm on $C_c(G)$. In fact, $\|f\| \le \|f\|_I$, where $\|f\|_I$ is the maximum of $\sup_u \int |f(x)|\, d\lambda^u(x)$ and $\sup_u \int |f(x^{-1})|\, d\lambda^u(x)$. The completion of $C_c(G)$ in $\|\cdot\|$, then, is a C^*-algebra, denoted by $C^*(G)$ or $C^*(G, \lambda)$, and is called the C^*-algebra of the groupoid G *(determined by λ).*

The proof will be given in the next chapter. It is a trivial consequence of Renault's disintegration theorem, Theorem 3.32. A straightforward computation shows that the quantity $\|\cdot\|_I$ is a norm on $C_c(G)$ such that $\|f^*\|_I = \|f\|_I$ and $\|f * g\|_I \le \|f\|_I \|g\|_I$. Thus, the completion of $C_c(G)$ in $\|\cdot\|_I$, denoted $L^I(G)$ or $L^I(G, \lambda)$, is a symmetric Banach algebra. Its enveloping C^*-algebra is $C^*(G)$.

EXAMPLES 2.43. (i) *Let the groupoid G be $X \times H$, where X is locally compact and H is a locally compact group, acting on X on the right. Fix a Haar measure λ_H on H and for $u \in G^{(0)}$ $(= X)$, set $\lambda^u = \epsilon_u \times \lambda_H$. Then, as we remarked in Remark 2.14 (referring there to left actions) it is straightforward to check that $\{\lambda^u\}_{u\in G^{(0)}}$ is a Haar system. Furthermore, for $f, g \in C_c(G)$ and $(y,t) \in G$, we have by definition,*

$$f * g(y,t) = \int f(x,s)g((x,s)^{-1}(y,t))\, d\lambda^{r(y,t)}(x,s)$$
$$= \int f(y,s)g((ys, s^{-1})(y,t))\, d(\epsilon_y \times \lambda_H(s))$$
$$= \int f(y,s)g(ys, s^{-1}t)\, d\lambda_H(s),$$

while $f^*(y,t) = \overline{f((y,t)^{-1})} = \overline{f(yt, t^{-1})}$. Thus, with respect to the operations, just defined, $C_c(G)$ is the algebra $C_c(X \times H)$ discussed in Chapter 1 *and $C^*(G)$ is $C^*(X,H)$. Note, however, that $\|\cdot\|_I$ does not coincide with the norm $\|\cdot\|_0$ defined in Chapter 1.*

EXAMPLE 2.44. (ii) *Let G be the trivial groupoid on a locally compact Hausdorff space X and let $\{\lambda^x\}_{x\in X}$ be the Haar system discussed earlier: $\lambda^x = \epsilon_x \times \lambda$, where λ is a measure on X with $\operatorname{supp}(\lambda) = X$. For $f, g \in C_c(G)$ and $(x,y) \in G$, we have*

$$\begin{aligned} f * g(x,y) &= \int f(u,v) g((u,v)^{-1}(x,y))\, d\lambda^{r(x,y)}(u,v) \\ &= \int f(x,v) g((v,x)(x,y))\, d(\epsilon_x x \lambda)(u,v) \\ &= \int f(x,v) g(v,y)\, d\lambda(v), \end{aligned}$$

and $f^*(x,y) = \overline{f((x,y)^{-1})} = \overline{f(y,x)}$. Thus the $*$-product is simply composition of the kernels determined by f and g and the adjoint is conjugate transposition. One expects, therefore, that in this case $C^*(G)$ is an elementary C^*-algebra (i.e., $C^*(G)$ is isomorphic to the compact operators on some Hilbert space). This, however, requires still some technology. A clearer understanding of the representations of $C^*(G)$ is necessary. The reader is encouraged to investigate for him or herself what might be required, by trying to prove that if X is a finite set of cardinality n, and if λ is given by counting measure, say, then $C^*(G)$ is isomorphic to $M_n(\mathbb{C})$.

We note in passing that when our analysis of this example is complete, we will have proved that $C^*(G)$ $(G = X \times X)$ is independent of λ. In one sense, this is a little surprising, but on reflection, it will prove to be quite natural. We note, too, that the norm $\|f\|_I$ in this context is simply the supremum of the L^1-norms of the rows and columns of the kernel determined by f.

We will have a great deal to say about general representations of $C_c(G)$ in the next chapter, but in advance of it, a special class deserves to be singled out. These representations serve as analogues of the regular representation of a group.

DEFINITION 2.45. *Let G be a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ and let μ be a (Radon) measure on $G^{(0)}$.*

(i) *We write $\nu = \mu \circ \lambda = \int \lambda^u \, d\mu$ for the measure on G defined by the formula $\int_G f(x)\, d\nu(x) = \int_{G^{(0)}} f(x)\, d\lambda^u(x)\, d\mu(u)$. We call ν the* measure induced *by μ, and we write ν^{-1} for the image of ν under the homeomorphism $x \to x^{-1}$.*

(ii) *For $f \in C_c(G)$, $\operatorname{Ind}\mu(f)$ is the operator on $L^2(\nu^{-1})$ defined by the formula*

$$\begin{aligned} \operatorname{Ind}\mu(f)\xi(x) &= \int f(y)\xi(y^{-1}x)\, d\lambda^{r(y)}(x) \\ &= f * \xi(x) \end{aligned}$$

(One checks with a little work that $\|\operatorname{Ind}\mu(f)\| \le \|f\|_I$ and that $\operatorname{Ind}\mu$ is a representation of $C_c(G)$ in the sense of Definition 2.41.) As the notation suggests, $\operatorname{Ind}\mu$ is called the induced representation *determined by (the multiplicity free representation of $C^*(G^{(0)}) = C_0(G^{(0)})$ associated with) μ.*

It is easy to check that in the case when the groupoid is a transformation group groupoid, then the definition of $\operatorname{Ind}\mu$ just given coincides with that in Example 1.7.

Although the details are a little tedious, it is not hard to show that $\operatorname{Ind}\mu$ is the direct integral $\int_{G^{(0)}}^{\oplus} \operatorname{Ind}(\epsilon_u)\,d\mu(u)$ and $\operatorname{Ind}\mu$ is a faithful representation of $C_c(G)$, if $\operatorname{supp}(\mu) = G^{(0)}$. Note, however, that a representation may be faithful on $C_c(G)$ without being faithful on $C^*(G)$. Indeed, if G is a nonamenable group, then there is essentially only one measure μ on $G^{(0)} = \{e\}$. In this case $\operatorname{Ind}\mu$ is the regular representation and has a nontrivial kernel.

The notion of amenability makes sense in the groupoid context, too, but until recently has proven to be more refractory than in the group theoretic setting. See [?]. We will discuss it at greater length in Chapter 6. For the moment, we are content to present the following definition.

DEFINITION 2.46. *The* reduced *C^*-algebra of the groupoid G, $C^*_{\mathrm{red}}(G)$ or$C^*_{\mathrm{red}}(G,\lambda)$, is defined to be the completion of $C_c(G,\lambda)$ in the norm*

$$\|f\|_{\mathrm{red}} = \sup\{\|\operatorname{Ind}\epsilon_u(f)\| \mid u \in G^{(0)}\}.$$

By the remarks preceding the definition, $\|f\|_{\mathrm{red}} = \|\operatorname{Ind}\mu(f)\|$ for any measure μ, with $\operatorname{supp}\mu = G^{(0)}$. Note, too, that for $\gamma \in G$, $\operatorname{Ind}\epsilon_{r(\gamma)}$ is unitarily equivalent to $\operatorname{Ind}\epsilon_{s(\gamma)}$. Indeed, translation by γ implements the desired unitary equivalence.

We record the following proposition for future reference. When specialized to the case when $G = \{1, 2, \cdots, n\}^2$, one sees that $\operatorname{Ind}\epsilon_u$ of $C^*_{\mathrm{red}}(G)$ is nothing more than the (left) representation of $M_n(\mathbb{C})$ on the column indexed by u.

PROPOSITION 2.47. *If G is the trivial groupoid on a locally compact Hausdorff space X and if $\{\lambda^x\}_{x\in X}$ is given by a measure λ with* $\operatorname{supp}(\lambda) = X$*, then $C^*_{\mathrm{red}}(G,\lambda)$ is isomorphic to $K(L^2(\lambda))$.*

PROOF. By definition, $\|f\|_{\mathrm{red}} = \sup_{u\in G^{(0)}}\{\|\operatorname{Ind}\epsilon_u(f)\|\}$. Since G is transitive, $\operatorname{Ind}\epsilon_u$ is unitarily equivalent to $\operatorname{Ind}\epsilon_v$ for all $u, v \in X$. Thus $\|f\|_{\mathrm{red}} = \|\operatorname{Ind}\epsilon_u(f)\|$, for any $u \in X$. But $\operatorname{Ind}\epsilon_u$ acts on $L^2(\nu^{-1})$, where $\nu = \epsilon_u \times \lambda$. So $\nu^{-1} = \lambda \times \epsilon_u$ and the map $W : L^2(\nu^{-1}) \to L^2(\lambda)$ defined by $(W\xi)(x) = \xi(x,u)$ is a Hilbert space isomorphism satisfying $W \operatorname{Ind}\epsilon_u(f) W^{-1} = \pi(f)$, where $\pi(f)$ is the operator on $L^2(\lambda)$ defined by the formula $\pi(f)\xi(x) = \int f(x,y)\xi(y)\,d\lambda(y)$. ☐

In the case when G is an r-discrete principal groupoid with a Haar system λ, one may scale λ so that it is given by counting measures, and then the algebraic operations on $C_c(R)$ are given by the formulae:

$$a * b(x,y) = \sum_z a(x,z)b(z,y)$$
$$a^*(x,y) = \overline{a(y,x)}.$$

That is, elements in $C_c(R)$ and, by extension, in $C^*(R)$ may be regarded as *matrices* indexed by R. (Just as in the case of transformation groups, without some sort of hypothesis relating to amenability, elements in $C^*(R)$ may be viewed as matrices only at a formal level.) We will see in Chapter 8 that C^*-algebras representable essentially as $C^*(R)$, for a suitable amenable r-discrete R, may be given an intrinsic characterization. We regard this as a fundamental coordinatization result.

CHAPTER 3

Representations of Groupoids

Our main objective in this chapter is to prove Renault's disintegration theorem [**171**]. As we indicated in the preface, our proof will be complete except for two details, which we discuss later. The first section is devoted to outlining the theory of Borel Hilbert bundles; i.e., direct integral theory. Much of the material here is well known. The difference between what we present and what is found, say, in Dixmier's treatise [**55**], is simply a matter of emphasis. Our presentation and notation follow Ramsay's articles [**157**] and [**159**] fairly closely. In the second section, we develop what we need about quasi-invariant measures. The third section is devoted to Renault's disintegration theorem and some of its immediate applications.

1. Borel Hilbert Bundles

The reader should keep in mind the perspective we are emphasizing in these notes: Groups are represented on Hilbert spaces, groupoids are represented on Hilbert bundles. The bundle concept is a familiar one, even if it is not immediately recognized as such; it appears in many guises. Set theoretically, a bundle of vector spaces, say, is simply the disjoint union of a family of vector spaces. Suppose, to be specific, that $\mathcal{V} = \{\mathcal{V}(x)\}_{x\in X}$ is a family of vector spaces indexed by a set X. We don't assume at the outset that $\mathcal{V}$ is disjoint. However, to "make them disjoint" we form the set $X*\mathcal{V} := \{(x,\xi) \mid \xi \in \mathcal{V}(x)\}$. Of course, then, $X*\mathcal{V}$ is the disjoint union of the spaces $\{x\}\times\mathcal{V}(x)$ which may be and often are identified with $\mathcal{V}(x)$. From time to time, it will be convenient to make this identification without fanfare to lighten the notation. We shall do this freely later on, but in this chapter we shall be fairly meticulous about the notation. Let $\pi : X*\mathcal{V} \to X$ be the natural projection, $\pi(x,\xi) = x$. We call $X*\mathcal{V}$ or the pair, $(X*\mathcal{V},\pi)$, a (vector) *bundle* over X. For each x in X, the space $\mathcal{V}(x)$, which we identify with $\pi^{-1}(x) = \{x\}\times\mathcal{V}(x)$, is called the *fibre over* x. Note that when the $\mathcal{V}(x)$ all coincide with a fixed space V, then $X*\mathcal{V}$ is simply $X\times V$.[1] Such a bundle is called trivial. The fibre over each point is a copy of V. A *section* of the bundle or a *vector field* is simply a function $f : X \to X*\mathcal{V}$ such that $\pi(f(x)) = x \in X$. Sections of the bundle $X*\mathcal{V}$ are closely linked to elements of $\Pi_{x\in X}\mathcal{V}(x)$; i.e. to functions $\phi : X \to \cup_{x\in X}\mathcal{V}(x)$ such that $\phi(x) \in \mathcal{V}(x)$, for all $x \in X$. Indeed, given a section f, we may write $f(x) = (x,\hat{f}(x))$, for a uniquely determined element $\hat{f} \in \Pi_{x\in X}\mathcal{V}(x)$; and given an element $f \in \Pi_{x\in X}\mathcal{V}(x)$, defining $\check{f}(x) := (x, f(x))$, determines a section. Because of this close link between sections of $X*\mathcal{V}$ and elements of $\Pi_{x\in X}\mathcal{V}(x)$, we will often abuse notation and write $f(x)$ for $(x,\hat{f}(x))$. However, when it is useful to make the distinction between sections and

[1]Throughout, we will use calligraphic letters, such as $\mathcal{V}$, to denote bundles. The fibre of $\mathcal{V}$ over x will then be denoted $\mathcal{V}(x)$. Individual spaces of the same sort as the fibers, but without reference to any particular bundle will be denoted with normal letters, e.g., V. Thus, throughout, $\mathcal{H}$'s denote Hilbert bundles, while H's denote Hilbert spaces.

elements of the product, we will continue to attach a hat ˆ to the latter. Of course, when $X * \mathcal{V} = X \times V$ is trivial, then a section f uniquely determines and is uniquely determined by a *function* $\hat{f} : X \to V$. Obviously, sections may be added pointwise and they may be multiplied by scalars. If the vector spaces $\mathcal{V}(x)$ have additional structure, then the sections inherit this structure. If, for example, each $\mathcal{V}(x)$ is an algebra, then the sections form an algebra under pointwise multiplication.

In analysis, one is interested in sections that are measurable, continuous, or smooth in some sense. These concepts require some structure on the bundles in question. Thus when discussing measurable sections one needs a Borel structure on the bundle; continuous sections require a topology, etc. The problem is that one may place a Borel structure or a topology on a bundle, but there is no reason *a priori* for there to be any sections that are measurable or continuous. In geometry, where one is interested in smooth bundles, one usually restricts attention to locally trivial bundles. However, in the setting of these notes, where infinite dimensional fibres are common and where there is no natural local trivialization in sight, it is convenient simply to prescribe the structure (measurable or topological) on the bundle by prescribing a set of sections. We are thus led to

DEFINITION 3.1. *An* analytic Borel Hilbert bundle *is vector bundle* $(X * \mathcal{H}, \pi)$, *where each space* $\mathcal{H}(x)$ *is a Hilbert space and where* $X * \mathcal{H}$ *is endowed with an analytic Borel structure such that the following axioms are satisfied:*

1. *A subset* E *in* X *is Borel if and only if* $\pi^{-1}(E)$ *is Borel.*
2. *There is a sequence* $\{f_n\}_{n=1}^{\infty}$ *of sections, called a* fundamental sequence, *such that*
   (a) *each function* $\tilde{f}_n : X * \mathcal{H} \to \mathbb{C}$, *defined by the formula* $\tilde{f}_n(x, \xi) = (f_n(x), \xi)_{\mathcal{H}(x)}$, *is Borel,*
   (b) *for each pair of fundamental sections,* f_n *and* f_m, *the function* $x \to (f_n(x), f_m(x))_{\mathcal{H}(x)}$ *is Borel, and*
   (c) *the functions* $\{\tilde{f}_n\}_{n=1}^{\infty}$ *together with* π *separate the points of* $X * \mathcal{H}$.

As is customary, we shall usually refer to an analytic Borel Hilbert bundle simply as a Hilbert bundle. Also, if the Borel structure happens to be standard, which usually will be the case, we may speak of a standard Borel Hilbert bundle. Later, we will have to consider topological Hilbert bundles and then we will have to be a bit more careful to distinguish among the various notions. Fortunately, one can usually tell from context which notion of Hilbert bundle one is considering. Also, as is customary, when dealing with a Hilbert bundle $X * \mathcal{H}$, we shall usually drop the subscript $\mathcal{H}(x)$ on the inner products. This should cause no difficulty.

We state for emphasis the following proposition, whose proof may be assembled from arguments in [**157**, p. 265] and [**159**, Section 1].

PROPOSITION 3.2. *Given the set* $X * \mathcal{H}$, *where* X *is an analytic Borel space and* $\mathcal{H}$ *is a family of Hilbert spaces indexed by* X, *and given a family of sections* $\{f_n\}_{n=1}^{\infty}$ *satisfying the conditions 2(b) and 2(c) of Definition 3.1, there is a unique Borel structure on* $X * \mathcal{H}$ *so that* $(X * \mathcal{H}, \pi)$ *becomes an analytic Borel Hilbert bundle and the sequence* $\{f_n\}_{n=1}^{\infty}$ *becomes a fundamental sequence for the bundle. If the Borel structure on* X *is standard, the same will be true of the induced Borel structure on* $X * \mathcal{H}$.

If $(X * \mathcal{H}, \pi)$ is a Hilbert bundle over X with a fundamental sequence $\{f_n\}_{n=1}^{\infty}$, then it is not hard to see that a section $f : X \to X * \mathcal{H}$ is Borel if and only if the function on X, $x \to (f(x), f_n(x))$, is Borel for each n. Note, too, that for each $x \in X$, the fibre $\mathcal{H}(x)$ is spanned by the vectors $f_n(x)$. (This is a consequence of condition 2(c) in Definition 3.1. We write $B(X * \mathcal{H})$ for the space of all Borel sections for the Hilbert bundle $(X*\mathcal{H}, \pi)$. The uniqueness assertion of the preceding paragraph means, in particular, that the Borel structure on a Hilbert bundle is independent of the particular fundamental sequence used provided, of course, that each term in one sequence is Borel with respect to the structure defined by the other and vice versa. The space $B(X * \mathcal{H})$ evidently is a module over the space $B(X)$ of all complex-valued Borel functions on X:

$$(\varphi \cdot f)(x) = \varphi(x)f(x) = (x, \varphi(x)\hat{f}(x)),$$

$\varphi \in B(X)$, and $f \in B(X * \mathcal{H})$. It is useful to note that one may apply the Gram-Schmidt orthogonalization process pointwise to a fundamental sequence $\{f_n\}_{n=1}^{\infty}$ and arrange to have the non-zero vectors in $\{f_n(x)\}_{n=1}^{\infty}$ form an orthonormal basis for $\mathcal{H}(x)$, for each x in X. Using such an "orthonormalized" fundamental sequence and Parseval's identity, it is easy to prove that for every Borel section f, the real-valued function $x \to \|f(x)\|$ is Borel. In summary, the space $B(X*\mathcal{H})$ satisfies the conditions place on the space $\mathcal{S}$ in Definition 1 of [**55**, II.1.3].

DEFINITION 3.3. *Given a Hilbert bundle* $(X*\mathcal{H}, \pi)$ *and a measure* μ *on* X, *we write* $L^2(\mu, \mathcal{H})$ *or* $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ *for* $\{f \in B(X * \mathcal{H}) | \int \|f(x)\|^2\, d\mu(x) < \infty\}$ *and call this space the* direct integral *of* $X*\mathcal{H}$ *or the* space of square integrable sections of $X * \mathcal{H}$.

Of course this space is a Hilbert space. What we are presenting, really, is nothing more than the concept of direct integral presented, say, in [**55**, Chapter II]. We are simply adding the emphasis on the Borel structure on $X * \mathcal{H}$.

Some examples are in order. First, of course, there are the trivial bundles, i.e., bundles of the form $X \times H$, where now H is a single Hilbert space. The Borel structure is the product Borel structure, where H is given the Borel structure determined by either the weak or norm topology - for separable spaces, these two are the same. Sections of this bundle, then, are nothing other than H-valued functions. A fundamental sequence is nothing but a point separating sequence of H-valued functions on X such that the values at each x generate H. More generally, suppose the space X is decomposed into the countable disjoint union of nonempty Borel subsets, $X = \cup X_n$, and suppose that for each n there is given a Hilbert space H_n. Let $X * \mathcal{H} = \cup(X_n \times H_n)$ with the obvious Borel structure and fundamental sequence. Then it is not hard to see that this $X * \mathcal{H}$ is a Hilbert bundle. In a sense that will be made precise in a moment, every Hilbert bundle is isomorphic to one of this kind. Thus, one might think that the theory is trivial; that it is unnecessarily laden with heavy notation. The point that one should keep in mind, however, is that Hilbert bundles are often presented in ways where it is not clear how to "trivialize" them in the fashion just discussed, and even when it is, the trivialization may not have anything to do with context in which the direct integral arose.

To illustrate this, we consider one of the most common ways that Hilbert bundles and their direct integrals arise: through the process of disintegrating a measure. This will be very useful as we proceed and so we pause to present a result that is one in a long line of decomposition theorems. (See [**53, 90, 141, 180**] and the discussion in Section 11 of [**115**].) The formulation we give here is essentially

Hahn's variation [**87**, Theorem 2.1] of a result due to Effros [**61**, Lemma 4.4]. (See also Théorème 2 in [**24**, Sect. 3, No. 3].) First, a definition.

DEFINITION 3.4. *Let $\pi : X \to Y$ be a Borel surjection between Borel spaces X and Y. A family of measures $\alpha = \{\alpha_y\}_{y\in Y}$ indexed by Y is called a* Borel π-system *or a* Borel system of measures for π, *in case for each $y \in Y$, $\operatorname{supp}\alpha_y \subseteq \pi^{-1}(y)$ and for each non-negative function f on X, the function $\alpha(f)(y) := \int f(x)\,d\alpha_y(x)$ is a measurable function on Y.*

REMARKS 3.5. *We are interested in this monograph only in the case when X and Y are countably generated Borel spaces (as mentioned in the prefeace) and when the measures α_y are sigma finite. This assumption will tacitly be in force whenever such systems are discussed. In fact, we shall often assume that our π-systems satisfy a stronger condition called properness. A Borel π-system α is* proper *in case there is a non-negative function f such that $\alpha(f) \equiv 1$.*

Examples of such systems are easy to come by. Of course, every Haar system is a Borel r-system (a fact that causes the English speaking world a little discomfort.) Haar systems are, in fact, continuous and the general notion of continuous π-systems make sense, when $\pi : X \to Y$ is a continuous map between topological spaces. We will discuss these further in Section 4 of Chapter 5.

THEOREM 3.6. *Let (X, λ) be an analytic Borel probability space and let ν be a σ-finite measure on X which is equivalent to λ. Let Y be a countably generated Borel space and suppose that $\pi : X \to Y$ is a Borel surjection. Suppose $\mu = \pi(\lambda)$ and set $P = d\nu/d\lambda$. Then*

I. *There is a Borel π-system $\nu = \{\nu_y\}_{y\in Y}$, where each ν_y is σ-finite, such that for each non-negative Borel function f on X, $\int f d\nu = \int \left(\int f d\nu_y\right) d\mu(y)$.*
II. *The map $y \to \nu_y$ is uniquely determined up to a μ-null set.*
III. *If $\{\lambda_y\}_{y\in Y}$ is the π-system determined by λ, then μ-almost every λ_y is a probability measure. Moreover, in this case, ν_y is equivalent to λ_y for μ-almost all y and for each y outside a μ-null set, the restriction of P to $\pi^{-1}(y)$ is a version of the Radon-Nikodym derivative $d\nu_y/d\lambda_y$.*

DEFINITION 3.7. *In the notation of the theorem, we write $\nu = \int \nu_y d\mu(y)$ and we call the triple $(\{\nu_y\}_{y\in Y}, \mu, \pi)$ the* π-decomposition of ν.

We remark in passing that it is not necessary to have μ be the image of a probability measure on Y. One can replace μ by an equivalent σ-finite measure, but then one must multiply each ν_y by a suitable scalar. Once μ and ν are fixed, the ν_y are uniquely determined modulo a μ-null set.

EXAMPLE 3.8. *In the setting of the theorem, let $\mathcal{H}(y) = L^2(\nu_y)$ to obtain a bundle $Y * \mathcal{H}$ over Y. The Borel structure on $Y * \mathcal{H}$ is given by a sequence of sections $\{f_n\}_{n=1}^\infty$ defined as follows. Choose a sequence of bounded, non-negative, Borel functions $\{g_n\}_{n=1}^\infty$ on X that separate points. Define $f_n : X \to Y * \mathcal{H}$ by the equation $f_n(y) = (y, \hat{f}_n(y))$, where $\hat{f}_n(y; x) = g_n(x)P^{-\frac{1}{2}}(x)$, $x \in \pi^{-1}(y)$.[2] Using Theorem 3.6, it is easy to check that conditions 2(b) and 2(c) of Definition 3.1 are satisfied by this sequence of sections, and so by Proposition 3.2, there is a unique*

[2]If $f : A \mapsto B$ is function and if, for each $a \in A$, $f(a)$ is itself a function on a set Ω_a then we shall separate the two independent variables by a semicolon and write $f(a;\omega)$ to denote the values of the function $f(a)$. While this notation may appear a little cumbersome at first glance, we have found it useful for clarifying the rolls that certain constructs play in the theory.

analytic Borel structure on $Y * \mathcal{H}$ *making it a Hilbert bundle. Moreover, the map* $W : L^2(X,\mu) \to \int_Y^{\oplus} \mathcal{H}(y)\,d\mu(y)$, *defined by the formula*

$$(W\xi)(y;x) = \begin{cases} \xi(x), & x \in \pi^{-1}(y) \\ 0 & x \notin \pi^{-1}(y) \end{cases},$$

is a Hilbert space isomorphism. This bundle may be written as the disjoint union of trivial bundles, but the process requires that rather arbitrary (but measurable) choices be made and there is no canonical or otherwise preferable way to do so.

EXAMPLE 3.9. *As another example of a Hilbert bundle that will be of use to us, but which cannot usually be trivialized in any natural way, consider a locally compact groupoid* G *with a Haar system* $\{\lambda^u\}_{u\in G^{(0)}}$. *Let* $\mathcal{H}(u) = L^2(\lambda^u)$, $u \in G^{(0)}$, *to get the bundle that we shall denote* $G^{(0)} * L^2(\lambda)$. *The Borel structure on* $G^{(0)} * L^2(\lambda)$ *is determined by a sequence of sections* $\{\xi_n\}_{n=1}^{\infty}$ *defined as follows. Choose a point-separating sequence of functions* $\{f_n\}_{n=1}^{\infty}$ *in* $C_c(G)$ *and define* $\xi_n : G^{(0)} \to G^{(0)} * L^2(\lambda)$ *by the formula*

$$\hat{\xi}_n(u;x) = f_n(x),\ x \in G^u.$$

One checks easily using the properties of the Haar system that conditions 2(b) and 2(c) of Definition 3.1 are satisfied so that there is a unique analytic Borel structure on $G^{(0)} * L^2(\lambda)$ *— in fact, it is standard — making it a Hilbert bundle over* $G^{(0)}$. *For reasons that will be made clear shortly, this bundle will be that associated to the left regular representation of* G.

Given Hilbert bundles $X_i*\mathcal{H}_i$, $i = 1,2$, the notion of a bundle map from $X_1*\mathcal{H}_1$ to $X_2 * \mathcal{H}_2$ makes perfectly good sense: It is a pair (τ, T), where $\tau : X_1 \to X_2$ is a map and $T : X_1 * \mathcal{H}_1 \to X_2 * \mathcal{H}_2$ is also a map, such that for each $x \in X_1$, there is a *linear* map $\hat{T}(x) : \mathcal{H}_1(x) \to \mathcal{H}_2(\tau(x))$ such that $T(x,\xi) = (\tau(x), \hat{T}(x)\xi)$ for all $(x,\xi) \in X_1 * \mathcal{H}_1$. We will want to assume that τ and T are Borel maps, of course. One usually says that the map T *covers* τ. We are particularly interested in the situation when $X_1 = X_2 := X$, say, and the bundle maps involved cover the identity map on X. We write the collection of all such maps as $\mathrm{Hom}(X * \mathcal{H}_1, X * \mathcal{H}_2)$. Evidently, this space may be identified with the Borel *sections* of the bundle $X*B(\mathcal{H}_1,\mathcal{H}_2) := \{(x,A) |\ x \in X,\ A \in B(\mathcal{H}_1(x),\mathcal{H}_2(x))\}$, where we give this bundle the Borel structure making the maps

$$(x,A) \to (Af_{1,n}(x), f_{2,m}(x))$$

all Borel, for any fundamental sequence $\{f_{i,n}\}_{n=1}^{\infty}$ for $X*\mathcal{H}_i$. Then the Borel structure on $X * B(\mathcal{H}_1,\mathcal{H}_2)$ is analytic and is standard if the Borel structure on each of the bundles is standard. A bounded Borel section of this bundle, A, gives rise to a *decomposable* operator from $\int_X^{\oplus} \mathcal{H}_1(x)\,d\mu(x)$ to $\int_X^{\oplus} \mathcal{H}_2(x)\,d\mu(x)$ which we also denote by A: $A\xi(x) = \hat{A}(x)\xi(x)$, for $\xi \in \int_X^{\oplus} \mathcal{H}_1(x)\,d\mu(x)$. These operators intertwine the multiplication representations of $L^\infty(X,\mu)$ on the two direct integrals. Conversely, as is well known (see [**10**] and [**55**]), every operator that intertwines these multiplication representations of $L^\infty(X,\mu)$ is given by a bounded Borel section of the bundle $X * B(\mathcal{H}_1,\mathcal{H}_2)$, i.e. a section A such that $\sup \|\hat{A}(x)\|$ is finite.

DEFINITION 3.10. *Given Hilbert bundles* $X * \mathcal{H}_i$, $i = 1,2$, *over the same space* X, *a* (fibre preserving) isomorphism *from* $X * \mathcal{H}_1$ *to* $X * \mathcal{H}_2$ *is a (Borel) section* V *of* $X * B(\mathcal{H}_1,\mathcal{H}_2)$ *such that* $\hat{V}(x)$ *is a Hilbert space isomorphism for each* x.

Two Hilbert bundles over the same space are called isomorphic *if there is a fibre preserving isomorphism between them.*

We have anticipated the following proposition earlier. The proof may be found in [**55, 159**].

PROPOSITION 3.11. *Given a Hilbert bundle $X * \mathcal{H}$, let $X_n = \{x | \dim(\mathcal{H}(x)) = n\}$, $n = 1, 2, \cdots, \infty$, and let H_n be a Hilbert space of dimension n. Then $X * \mathcal{H}$ is isomorphic to* $\bigcup_{n \in \{1,2,\cdots,\infty\}} X_n \times H_n$.

The sets X_n in the proposition are sometimes called *the sets of constant* or *uniform multiplicity* for the bundle. In Example 3.8, the set of constant multiplicity n is the set on which π is n to 1, provided this set is not null.

DEFINITION 3.12. *Given a Hilbert bundle $X * \mathcal{H}$, we write* $\mathrm{Iso}(X * \mathcal{H})$ *for* $\{(x, V, y) |\ V : H(y) \to H(x)$ is a Hilbert space isomorphism$\}$ *endowed with the weakest Borel structure so that the maps*

$$(x, V, y) \to (V f_n(y), f_m(x))$$

are Borel for every n and m, where $\{f_n\}_{n=1}^{\infty}$ is a fundamental sequence for $X * \mathcal{H}$.

Of course $\mathrm{Iso}(X * \mathcal{H})$ is a groupoid in the operations described in Example 2.2. Its unit space is naturally identified with X. In fact, $\mathrm{Iso}(X * \mathcal{H})$ is an analytic Borel groupoid in the sense of Definition 2.25, as is transparent from the easily proved

PROPOSITION 3.13. *If the Hilbert bundle $X * \mathcal{H}$ is Borel isomorphic to $\bigcup X_n \times H_n$, where X is the disjoint union of the X_n and H_n is a separable Hilbert space*[3]*, then* $\mathrm{Iso}(X * \mathcal{H})$ *is Borel isomorphic to $\bigcup X_n \times \mathcal{U}(H_n) \times X_n$, where $X_n \times \mathcal{U}(H_n) \times X_n$ is the Cartesian product of the trivial groupoid, $X_n \times X_n$, endowed with the product Borel structure, and $\mathcal{U}(H_n)$ — the Borel* group *of all unitary operators on H_n endowed with the Borel structure determined by the weak operator topology.*

2. Quasi-invariant Measures

We are almost at the point where we can define a representation of a groupoid G. It will certainly involve a Hilbert bundle $G^{(0)} * \mathcal{H}$ over the unit space $G^{(0)}$ of G and it will involve a Borel homomorphism of G into $\mathrm{Iso}(G^{(0)} * \mathcal{H})$. The one missing ingredient is the notion of a quasi-invariant measure on $G^{(0)}$. In order to make this precise and to develop a few useful properties of quasi-invariant measures, we need first to say a few words about "fibred products of disintegrated measures". We follow [**157**].

Suppose that for $i = 1, 2$, we are given an analytic Borel measure space X_i with a σ-finite measure ν_i. Suppose, too, that we are given a third such measure space with probability measure (Y, μ), and Borel surjections $\pi_i : X_i \to Y$ that yield π-decompositions as defined in Definition 3.7, $(\{\nu_{i,y}\}_{y \in Y}, \mu, \pi_i)$. Thus $\nu_i = \int \nu_{i,y}\, d\mu(y)$, $i = 1, 2$. Form $X_1 * X_2 := \{(x_1, x_2) \in X_1 \times X_2 |\ \pi_1(x_1) = \pi_2(x_2)\}$. This is a standard Borel space and there is an obvious Borel surjection π mapping it onto Y. For each $y \in Y$, the measure $\nu_{1,y} \times \nu_{2,y}$ is a measure on $X_1 \times X_2$ with support in the set $\{(x_1, x_2) |\ \pi_1(x_1) = \pi_2(x_2) = y\}$. Furthermore, if f is a non-negative Borel function on $X_1 \times X_2$, and if $f^0(y)$ is defined to be $\int f\, d(\nu_{1,y} \times \nu_{2,y})$, then f^0 is a Borel function on Y. This is evident for functions that are products

[3] This need not be the decomposition of $X * \mathcal{H}$ in terms of the sets of uniform multiplicity.

of functions in each of the variables separately and so the assertion follows from linearity and the monotone convergence theorem. We define the measure $\nu_1 * \nu_2$ to be $\int \nu_{1,y} \times \nu_{2,y}\, d\mu(y)$; that is, for non-negative Borel functions f on $X_1 \times X_2$, $\int f\, d(\nu_1 * \nu_2) = \int f^0(y)\, d\mu(y)$, by definition. It is clear that $\nu_1 * \nu_2$ is supported on $X_1 * X_2$ and $(\{\nu_{1,y} \times \nu_{2,y}\}_{y\in Y}, \mu, \pi)$ is its π-decomposition by definition.

It is worthwhile to note, too, that if $\rho_i : X_1 * X_2 \to X_i$, $i = 1, 2$, is the canonical projection, $\rho_i(x_1, x_2) = x_i$, then ρ_i is a Borel map that is surjective because each π_i is surjective. We may therefore consider the decompositions of $\nu_1 * \nu_2$ with respect to each ρ_i. The result, proved in [**157**, p. 266] on the basis of Fubini's theorem, is

$$\begin{aligned}\nu_1 * \nu_2 &= \int \epsilon_{x_1} \times \nu_{2,\pi_1(x_1)}\, d\nu_1(x_1)\\ &= \int \nu_{1,\pi_2(x_2)} \times \epsilon_{x_2}\, d\nu_2(x_2),\end{aligned}$$

where, recall, ϵ_{x_i} denotes the point mass at x_i.

DEFINITION 3.14. *Let G be a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ and let μ be a (Radon) measure on $G^{(0)}$. If $\nu = \int \lambda^u\, d\mu(u) := \mu \circ \lambda$ is the induced measure on G (see Definition 2.45) and if ν^{-1} is its image under inversion, then we call the measure μ* quasi-invariant (q.i.) *in case ν and ν^{-1} are mutually absolutely continuous. In this case, we write Δ for the Radon-Nikodym derivative $d\nu/d\nu^{-1}$ and call it the* modular function *or* modulus *of μ. If the $\Delta \equiv 1$ a.e. ν, i.e., if $\nu = \nu^{-1}$, then we say that the measure μ is* invariant.

The reason for the terminology "modular function" is that Δ behaves very much like the modular function for a locally compact group. In particular, it is a *homomorphism* from G to the positive real numbers under multiplication $\mathbb{R}_+^\times$. Of course, this statement is a bit imprecise, since Δ is not uniquely determined. The precise statement follows, a proof will be given in Chapter 4, as a corollary of Theorem 4.16.

THEOREM 3.15. [**87**, Corollary 3.14]*Given a quasi-invariant measure μ on the unit space of a groupoid G with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, it is possible to choose the modular function of μ, Δ, to be a Borel* homomorphism *from G to $\mathbb{R}_+^\times$. Moreover, if μ' is another quasi-invariant measure on $G^{(0)}$ that is equivalent to μ, so that $\mu' = g\mu$, for a suitable non-negative function g, and if Δ' is the modular function of μ', then $\Delta'(x) = g(r(x))\Delta(x)g(s(x))^{-1}$ a.e. ν, where ν is the measure induced by μ. In particular, μ is equivalent to an* invariant measure *if and only if $\Delta(x) = g(r(x))g(s(x))^{-1}$, a.e. ν, for some non-negative function g.*

EXERCISE 3.16. *If $G = X \times H$ is the transformation group groupoid determined by the action of a locally compact group H on the locally compact space X, as discussed in Chapter 1, and if $\lambda^x = \epsilon_x \times \lambda_H$, giving the Haar system discussed in Chapter 2, then a measure μ on $X = G^{(0)}$ is quasi-invariant in the sense of Definition 3.14 if and only if it is quasi-invariant in the sense of Chapter 1, Definition 1.10. Moreover, the modular function Δ of μ is given by the formula $\Delta = J \cdot \delta$ where J is the function defined in Chapter 1 and $\delta = d\lambda_H/d\lambda_H^{-1}$ is the modular function of H. Thus, if H is unimodular ($\delta \equiv 1$), then μ is invariant under the action of H if and only if it is invariant in the sense of Definition 3.14. (See* [**123**, p. 198] [**157**, Theorem 4.3] *and* [**87**, Example 3.16]*.)*

EXERCISE 3.17. *Suppose that X is a locally compact space and that G is the trivial groupoid $X \times X$. We give G the Haar system defined by the formula $\lambda^x = \epsilon_x \times \lambda$ for some prescribed measure λ on X with full support (see Chapter 2, Exercise 2.29). Then a measure μ on X is q.i. if and only if $\mu \sim \lambda$. In this case, $\nu = \mu \times \lambda$ and $\Delta(x,y) = g(x)/g(y)$, a.e. ν, where $g = d\mu/d\lambda$.*

REMARK 3.18. *The question immediately arises: For a general locally compact groupoid G with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, do quasi-invariant measures exist on $G^{(0)}$? The answer is yes. To see this, let μ_0 be any probability measure on $G^{(0)}$ and let $\nu_0 := \int \lambda^u \, d\mu(u)$. Then ν_0 is a σ-finite measure on G that is usually not finite. Let ν be a probability on G that is mutually absolutely continuous with ν_0 and define μ to be the image of ν under s. Then, in a rather straightforward fashion, one can prove that μ is quasi-invariant and, further, that if μ_0 is quasi-invariant to begin with, then μ_0 is equivalent to μ. The details may be found on pages 24 and 25 of* [**171**]. *The measure μ is called the* saturation *of μ_0. It is sometimes written $[\mu_0]$.*

In the language of Chapter 2, Definition 2.4, the modular function of a quasi-invariant measure is a *cocycle*, the modular functions of two equivalent measures are *cohomologous*, and a measure is equivalent to an invariant measure if and only if the modular function is a *coboundary.* Example 3.17 shows that the modular function of a quasi-invariant measure on a trivial groupoid is a coboundary. This is not too surprising, given Proposition 2.5. In fact, the proof presented there works here, too.

PROPOSITION 3.19. *If $G_1 = X \times X$ is the trivial analytic Borel groupoid determined by an analytic Borel space X, and if $\phi : G_1 \to G_2$ is a Borel homomorphism from G_1 to an analytic Borel groupoid G_2, then ϕ is a Borel coboundary; that is, there is a Borel map $b : X \to G_2$ such that $\phi(x,y) = b(x)b(y)^{-1}$.*

PROOF. The point is that since ϕ is Borel, so is b, defined by the formula, $b(x) = \phi(x, y_0)$, for any prescribed y_0. □

The problem with cocycles in the Borel setting is the intervention of null sets. We discussed this some in Chapter 1 and we will come to grips with it more thoroughly in Chapter 4. For now, we present the theorem, due to Ramsay [**157, 161**], that allows us to sweep most of the difficulties caused by null sets aside. To state it, we need a bit more notation. Let G be a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Consider the set of composable pairs from G, $G^{(2)} = \{(x,y) \in G \times G \mid s(x) = r(y)\}$. This is the fibred product of G with itself determined by r and s. If μ is a measure on $G^{(0)}$, with induced measure $\nu = \mu \circ \lambda$ on G, then the fibred product $\nu^{(2)}$ is the measure on $G^{(2)}$ with decomposition

$$\nu^{(2)} = \int \lambda_u \times \lambda^u \, d\mu(u),$$

where λ_u is the image of λ^u under inversion. (Thus, in particular, the support of λ_u is G_u.) It might be preferable to denote $\nu^{(2)}$ by $\nu^{-1} * \nu$, given the earlier notation. However, the notation $\nu^{(2)}$ is the one most commonly found in the literature.

The following theorem will be proved as Theorem 4.16 in Chapter 4. The proof is given in two steps. First one proves [**157**, Theorem 5.1], and then [**161**, Theorem 3.2]. We state the result here for the purpose of reference and emphasis.

THEOREM 3.20. [**157**, Theorem 5.1][**161**, Theorem 3.2] *Let G be a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ and let H be an analytic Borel groupoid. Let μ be a quasi-invariant measure with induced measure $\nu = \mu \circ \lambda$. If $\phi_0 : G \to H$ is a Borel map such that*

$$\nu^{(2)}\left(\{(x, y) \in G^{(2)} \mid \phi_0(x)\phi_0(y) \neq \phi_0(xy)\}\right) = 0,$$

then there is a Borel homomorphism $\phi : G \to H$ such that $\phi_0 = \phi$ a.e. ν.

A map of the form ϕ_0 is called an *a.e. homomorphism.* Thus, an a.e. homomorphism is almost everywhere equal to a homomorphism. The proof rests, in part, on the following lemma, also due to Ramsay [**157**, Lemma 5.2], which we shall prove in Chapter 4, Lemma 4.9, and which will be used explicitly in our proof of Renault's disintegration theorem.

LEMMA 3.21. *Let μ be a quasi-invariant measure on $G^{(0)}$ and let Σ be a subset of G that is closed under multiplication and that contains a ν-conull subset of G, then there is a conull Borel subset U of $G^{(0)}$ such that the reduction $G|_U \subseteq \Sigma$.*

3. Renault's Disintegration Theorem

A Hilbert space representation of a groupoid must take place in the isomorphism groupoid of a Hilbert bundle in the sense of

DEFINITION 3.22. *Let G be a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. A* representation *of G (and λ) is a triple, $(\mu, G^{(0)} * \mathcal{H}, L)$ where μ is a quasi-invariant measure on $G^{(0)}$, $G^{(0)} * \mathcal{H}$ is a Hilbert bundle over $G^{(0)}$, and $L : G \to \mathrm{Iso}(G^{(0)} * \mathcal{H})$ is a Borel homomorphism that preserves the unit space $G^{(0)}$ in the sense that $L(x) = (r(x), \hat{L}(x), s(x))$, where $\hat{L}(x) : \mathcal{H}(s(x)) \to \mathcal{H}(r(x))$ is a Hilbert space isomorphism.*

REMARK 3.23. *Quite frequently, when studying representations of groupoids, one first encounters or produces an a.e. representation, i.e., an a.e. homomorphism of a groupoid into the isomorphism groupoid of a Hilbert bundle. This will be the case, in particular, in the proof of the Renault's disintegration theorem, Theorem 3.32. An a.e. representation, then, consists of a quasi-invariant measure μ, a Hilbert bundle $G^{(0)} * \mathcal{H}$ on $G^{(0)}$, a conull subset U of $G^{(0)}$, and a Borel map $L : G|_U \to \mathrm{Iso}(G^{(0)} * \mathcal{H}|_U)$, where $G^{(0)} * \mathcal{H}|_U$ is just the restriction of $G^{(0)} * \mathcal{H}$ to U, such that*

1. *$\hat{L}(x) : \mathcal{H}(s(x)) \to \mathcal{H}(r(x))$ is a Hilbert space isomorphism and $L(x) = (r(x), \hat{L}(x), s(x))$ for ν-almost all $x \in G|_U$.*
2. *$\hat{L}(u) = I_u$, the identity operator on $\mathcal{H}(u)$, for μ-almost all $u \in U$.*
3. *$L(x)L(y) = L(xy)$, a.e. $\nu^{(2)}$, and*
4. *$L(x)^{-1} = L(x^{-1})$, a.e. ν.*

By Theorem 3.20 there is a representation that agrees with L almost everywhere with respect to ν.

If μ is a quasi-invariant measure on $G^{(0)}$ and if U is a conull Borel set in $G^{(0)}$, then $G|_U$ is called an *inessential contraction* or an *inessential reduction* of G. As we shall see, when dealing with the measure theoretic aspects of groupoids, inessential contractions appear quite frequently. For our immediate purposes, they play a role in the notion of equivalence of representations.

DEFINITION 3.24. *Two representations* $(\mu_i, G^{(0)} * \mathcal{H}_i, L_i)$, $i = 1, 2$, *are called* equivalent *if and only if* $\mu_1 \sim \mu_2$ *and there is a fibre preserving isomorphism*

$$V : (G^{(0)} * \mathcal{H}_1)|_U \to (G^{(0)} * \mathcal{H}_2)|_U,$$

where U *is a conull subset of* $G^{(0)}$, *such that*

$$\hat{V}(r(x))\hat{L}_1(x) = \hat{L}_2(x)\hat{V}(s(x))$$

for $x \in G|_U$.

From the perspective of Definition 4.10 in the next chapter, equivalent representations really ought to be called weakly equivalent. However, the terminology just presented has become accepted.

PROPOSITION 3.25. *Let* $(\mu, G^{(0)} * \mathcal{H}, L)$ *be a representation of the locally compact groupoid* G *with Haar system* $\{\lambda^u\}_{u \in G^{(0)}}$. *For* $f \in C_c(G)$ *and* ξ *and* η *in* $\int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\mu(u)$ *the equation,*

$$(L(f)\xi, \eta) = \int f(x)(\hat{L}(x)\xi(s(x)), \eta(r(x)))\, d\nu_0(x),$$

where $\nu_0 := \Delta^{-\frac{1}{2}}\nu$, *defines a representation of* $C_c(G)$ *on* $\int_{G^{(0)}}^{\oplus} \mathcal{H}(x)\, d\mu(x)$ *and the inequality,* $\|L(f)\| \le \|f\|_I$, *is satisfied for all* $f \in C_c(G)$. *Moreover, equivalent representations of* G *yield unitarily equivalent representations of* $C_c(G)$.

PROOF. The proof is straightforward and the details may be found in [**171**, Proposition II.1.7]. We include only the proof of the boundedness to show why the "symmetrized" measure ν_0 is used. Evidently, $|f(x)(L(x)\xi(s(x)), \eta(r(x)))| \le |f(x)|\, \|\xi(s(x))\|\, \|\eta(r(x))\|$. So the expression defining $(L(f)\xi, \eta)$ is dominated in absolute value by

$$\int |f(x)|\, \|\xi(s(x))\|\, \|\eta(r(x))\|\; d\nu_0(x) = \int \left(\Delta^{-\frac{1}{2}}(x)\, \|\xi(s(x))\|\, \|\eta(r(x))\|\right) |f(x)|\; d\nu(x).$$

Applying the Cauchy-Schwarz inequality, we see that this is dominated, in turn, by

$$\left(\int \Delta^{-1}(x)\, \|\xi(s(x))\|^2\, |f(x)|\; d\nu(x)\right)^{\frac{1}{2}} \left(\int \|\eta(r(x))\|^2\, |f(x)|\; d\nu(x)\right)^{\frac{1}{2}}.$$

Using the facts that $\nu = \mu \circ \lambda$ and $\nu^{-1} = \Delta^{-1} d\nu$ this product may be rewritten as

$$\left(\int |f(x)|\; d\lambda_u(x)\, \|\xi(u)\|^2\, d\mu(u)\right)^{\frac{1}{2}} \left(\int |f(x)|\; d\lambda^u(x)\, \|\eta(u)\|^2\, d\mu(u)\right)^{\frac{1}{2}},$$

which, from the definition of the I-norm, is dominated by

$$\left(\|f\|_I^{\frac{1}{2}}\, \|\xi\|\right) \cdot \left(\|f\|_I^{\frac{1}{2}}\, \|\eta\|\right) = \|f\|_I\, \|\xi\|\, \|\eta\|.$$

This completes the proof. □

DEFINITION 3.26. *The representation* L *of* $C_c(G)$ *determined by a representation* $(\mu, G^{(0)} * \mathcal{H}, L)$ *of* G *in Proposition 3.25 is called the* integrated form of $(\mu, G^{(0)} * \mathcal{H}, L)$ *and* $(\mu, G^{(0)} * \mathcal{H}, L)$ *is called the* disintegrated form of L.

Before turning to some examples, it is worthwhile to note that rather than expressing $L(f)$ through the sesquilinear form it determines, one may write it explicitly as an operator on $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ via the formula:

$$(L(f)\xi)(u) = \int f(x)\hat{L}(x)\xi(s(x))\Delta^{-\frac{1}{2}}(x)\, d\lambda^u(x), \text{a.e. } \mu,$$

$f \in C_c(G)$, $\xi \in \int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$. Furthermore, if $G^{(0)} * \mathcal{H}$ is isomorphic to the constant bundle $G^{(0)} \times H$, say by V: $V(u, \xi) = (u, \hat{V}\xi)$, and if $L_0 : G \to \mathcal{U}(H)$ is defined by the formula, $L_0(x) = \hat{V}(r(x))\hat{L}(x)\hat{V}(s(x))^{-1}$, then V, regarded as a Hilbert space isomorphism from $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ onto $L^2(\mu, H)$, gives rise to the equation

$$(VL(f)V^{-1}\xi)(u) = \int f(x)L_0(x)\xi(s(x))\Delta^{-\frac{1}{2}}(x)\, d\lambda^u(x),$$

$f \in C_c(G)$, $\xi \in L^2(\mu, H)$. The point is that all the vectors come from and are added up in the same space H.

EXAMPLE 3.27. *Suppose the groupoid G is the transformation group groupoid determined by the action of a locally compact group H on a locally compact space X, as in Chapter 1, and identify $G^{(0)}$ with X. Suppose the Haar system is given by the equation $\lambda^x = \epsilon_x \times \lambda_H$, as in Example 2.43. Recall that $r(x,t) = x$, and that $s(x,t) = xt$, from Example 2.2. Suppose we are given a representation of G, $(\mu, G^{(0)} * \mathcal{H}, L)$ and that $G^{(0)} * \mathcal{H}$ is isomorphic to the constant bundle $G^{(0)} \times H_0$. Then the map $L_0 : G\ (= X \times H) \to \mathcal{U}(H_0)$ described above is a strict cocycle. Recall from Exercise 3.16 that Δ is given by the formula, $\Delta(x,t) = J(x,t)\delta(t)$, where $\delta = d\lambda_H/d\lambda_H^{-1}$. Recall, too, that $J^{-\frac{1}{2}}(x,t) = J^{\frac{1}{2}}(xt, t^{-1})$. Consequently, the representation L of $C_c(G)$, gotten through integration and expressed in terms of L_0 as above, is given by the formula*

$$VL(f)V^{-1}\xi(x) = \int f(x,t)L_0(x,t)\xi(xt)J^{\frac{1}{2}}(xt, t^{-1})\delta^{-\frac{1}{2}}(t)\, d\lambda_H(t)$$

$f \in C_c(X \times H)$, $\xi \in L^2(\mu, H_0)$. Except for the factor of $\delta^{-\frac{1}{2}}$, this is precisely the integrated form of the representation described in Theorem 1.14. The presence of $\delta^{-\frac{1}{2}}$ may be accounted for by our elimination of the modular function in formula for the adjoint on $C_c(X \times H)$.

EXAMPLE 3.28. *In this example, G is an arbitrary locally compact groupoid with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$ and μ is an arbitrary (Radon) measure on $G^{(0)}$. We show how to write $L :=$ Ind μ in integrated form using Theorem 3.6. (Recall Definition 2.45.)*

The representation L acts on the Hilbert space $L^2(\nu^{-1})$ according to the formula

$$(L(f)\xi, \eta) = (f * \xi, \eta) = \int f * \xi(y)\overline{\eta(y)}\, d\nu^{-1}(y).$$

If one changes variables in the integral and expands the convolution, the integral becomes

$$\int\int f(x)\xi(x^{-1}y^{-1})\, d\lambda^{s(y)}(x)\overline{\eta(y^{-1})}\, d\nu(y) = \int\int f(x)\tilde{\xi}(yx)\, d\lambda^{s(y)}(x)\overline{\tilde{\eta}(y)}\, d\nu(y),$$

where $\tilde{\xi}(y) = \xi(y^{-1})$. Apply Theorem 3.6 to write the s-decomposition of ν as $\nu = \int \nu_u\, d[\mu](u)$, where $[\mu]$ is the image under s of a probability measure on G

equivalent to ν. *Thus, each* ν_u *is supported on* $G_u(= s^{-1}(u))$. *Then, as was noted in Remark 3.18,* $[\mu]$ *is quasi-invariant by Proposition I.3.6 of* **[171]**. *Inserting this decomposition into the last integral, we obtain*

$$\int\int\int f(x)\tilde{\xi}(yx)\,d\lambda^{s(y)}(x)\overline{\tilde{\eta}(y)}\,d\nu_u(y)\,d[\mu](u).$$

Since $s(y) = r(x) = u$ *in this integral, we may apply Fubini's theorem and write it as*

$$\iiint f(x)\tilde{\xi}(yx)\overline{\tilde{\eta}(y)}\,d\nu_u(y)\,d\lambda^u(x)\,d[\mu](u)$$
$$= \int f(x)\left[\int \tilde{\xi}(yx)\overline{\tilde{\eta}(y)}\,d\nu_{r(x)}(y)\right]\,d[\nu](x),$$

where $[\nu] = \int \lambda^u\,d[\mu](u)$. *If* ν^u *is the image of* ν_u *under inversion, then the integral in the brackets becomes*

$$\int \xi(x^{-1}y)\overline{\eta(y)}\,d\nu^{r(x)}(y).$$

Since $[\mu]$ *is quasi-invariant,* $\Delta := d[\nu]/d[\nu]^{-1}$ *may be chosen to be a Borel homomorphism by Theorem 3.15. So, if we let* $G^{(0)} * \mathcal{H}$ *be the Hilbert bundle associated with the disintegration of* $[\nu]^{-1}$ *with respect to* $r : G \to G^{(0)}$ *(see Example 3.8), then* $\mathcal{H}(u) = L^2(\nu^u)$ *and we obtain a representation* L *of* G *on* $G^{(0)} * \mathcal{H}$ *through the formula*

$$(\hat{L}(x)\xi(s(x)))(y) = \xi(x^{-1}y)\Delta^{\frac{1}{2}}(x),$$

$y \in G^{r(x)}$. *With this notation, we conclude that*

$$(L(f)\xi,\eta) = \int\left[\int \xi(x^{-1}y)\overline{\eta(y)}\,d\nu^{r(x)}(y)\right]\,d[\nu](x)$$
$$= \int (\hat{L}(x)\xi(s(x)),\eta(r(x)))\,d[\nu]_0(x),$$

where $[\nu]_0 = \Delta^{-\frac{1}{2}}[\nu]$.

DEFINITION 3.29. *Let* μ *be a quasi-invariant measure on* $G^{(0)}$. *The (left) regular representation of* G *is the representation* $(\mu, G^{(0)} * L^2(\lambda), L)$ *where*

$$\hat{L}(x) : L^2(\lambda^{s(x)}) \to L^2(\lambda^{r(x)})$$

is defined by the formula

$$(\hat{L}(x)\xi(s(x)))(y) = \xi(x^{-1}y).$$

The integrated form of this representation is called the (left) regular representation *of* $C_c(G)$ *(also, of* $C^*(G,\lambda)$*) on* μ.

EXERCISE 3.30. *Let* μ *be a quasi-invariant measure on* $G^{(0)}$, *let* ν *be the induced measure on* G, *and let* L *be the regular representation of* $C_c(G)$ *on* μ. *The map* $W : L^2(\nu) \to L^2(\nu^{-1})$ *defined by the formula,* $W\xi = \xi \cdot \Delta^{\frac{1}{2}}$ *is a Hilbert space isomorphism that implements a unitary equivalence between* L *and* Ind μ. *(See* **[171**, II.1.10]*.)*

We require the following lemma. While the proof could be given here, it seems preferable to prove it in a more general context later (See Chapter 5, Proposition 5.39).

LEMMA 3.31. *There is a self-adjoint approximate identity* $\{e_n\}_{n=1}^{\infty}$ *for* $C_c(G)$ *in the inductive limit topology.*[4]

We come now to the primary objective of this chapter,

THEOREM 3.32. [**174**, Proposition 4.2] *Let* H *be a Hilbert space and let* H_0 *be a dense linear subspace. Suppose that* L *is a homomorphism from* $C_c(G)$ *into the algebra of linear transformations on* H_0 *satisfying the following three conditions:*

(a) *L is non-degenerate in the sense that the span of* $\{L(f)\xi \mid f \in C_c(G),\ \xi \in H_0\}$ *is dense in* H.
(b) *For each* $\xi, \eta \in H_0$, *the functional* $L_{\xi,\eta} : C_c(G) \to \mathbb{C}$, *defined by the equation* $L_{\xi,\eta}(f) = \langle \xi, L(f)\eta \rangle$, *is continuous with respect to the inductive limit topology on* $C_c(G)$.
(c) *For all* $f \in C_c(G)$ *and for all* $\xi, \eta \in H_0$, $\langle \xi, L(f^*)\eta \rangle = \langle L(f)\xi, \eta \rangle$.

Then each $L(f)$ *is bounded and so extends uniquely to an operator, also denoted by* $L(f)$, *on all of* H. *The map* $f \to L(f)$ *is a representation of* $C_c(G)$ *on* H *and there is a representation* $(\mu, G^{(0)} * \mathcal{H}, U)$ *such that* L *is unitarily equivalent to the integrated form of* $(\mu, G^{(0)} * \mathcal{H}, U)$.

PROOF. The idea of the proof is to try to identify H with a completion of $C_c(G) \otimes H_0$[5] and then to transfer the bundle structure implicit in $C_c(G)$ to $C_c(G) \otimes H_0$ by analyzing $C_c(G) \otimes H_0$ in terms of $\bigcup_{u \in G^{(0)}} C_c(G^u) \otimes H_0$. The map from $C_c(G) \times H_0$ to H_0 defined by (f, ξ) to $L(f)\xi$ is bilinear and has for its range the span[6] of $L(C_c(G))H_0$. We write H_{00} for this span. Thus H_{00} is a quotient of $C_c(G) \otimes H_0$. We want to carry the inner product structure on H_{00} back to $C_c(G) \otimes H_0$. To this end, define the sesquilinear form $\langle \cdot, \cdot \rangle$ on $C_c(G) \otimes H_0$ by the formula $\langle f \otimes \xi, g \otimes \eta \rangle = \langle \xi, L(f^* * g)\eta \rangle$. Then $\langle \cdot, \cdot \rangle$ is positive semi-definite:

$$\begin{aligned}\langle \sum_i f_i \otimes \xi_i, \sum_j f_j \otimes \xi_j \rangle &= \sum_{i,j} \langle \xi_i, L(f_i^* * f_j)\xi_j \rangle = \sum_{i,j} \langle \xi_i, L(f_i^*)L(f_j)\xi_j \rangle \\ &= \sum_{i,j} \langle L(f_i)\xi_i, L(f_j)\xi_j \rangle = \langle \sum_i L(f_i)\xi_i, \sum_j L(f_j)\xi_j \rangle \\ &= \| \sum_i L(f_i)\xi_i \|^2 \geq 0,\end{aligned}$$

and is zero only when $\sum L(f_i)\xi_i = 0$. Let $\mathcal{N} = \{\sum_i f_i \otimes \xi_i \mid \sum L(f_i)\xi_i = 0\}$ be the kernel of this seminorm and let K be the completion of $C_c(G) \otimes H_0/\mathcal{N}$ in the induced norm. The computations show that the map W from H_{00} to K defined by the formula $W(\sum L(f_i)\xi_i) = \sum f_i \otimes \xi_i + \mathcal{N}$ is well defined, isometric, and has range dense in K, of course. By hypothesis (a), W extends to a Hilbert space isomorphism from *all* of H onto K. We shall henceforth identify H with K via W and we shall identify H_{00} with $C_c(G) \otimes H_0/\mathcal{N}$ in K.

Hypothesis a) guarantees that H_{00} is dense in H. It is crucial for our analysis to know that, in fact, the span of $L(C_c(G))H_{00}$ is dense in H. To see this, first note that hypothesis b) implies that the functionals $L_{\xi,\eta}$ determine Radon measures on

[4] Note that a self-adjoint, one-sided approximate identity automatically is two-sided, since taking adjoints is a homeomorphism in the inductive limit topology on $C_c(G)$.

[5] Throughout the proof, the symbol $\otimes$ will denote simply the *algebraic* tensor product.

[6] The term "span" will always mean linear span. No closures are implied or taken unless explicitly stated.

G. We shall identify $L_{\xi,\eta}$ with the Radon measure it determines and continue to write $L_{\xi,\eta}(f) = (\xi, L(f)\eta)$, $\xi, \eta \in H_0$, $f \in C_c(G)$. This formula shows that $L_{\xi,\eta}(f)$ is linear in ξ and conjugate linear in η and f. Second, recall from Lemma 3.31 that $C_c(G)$ has a self-adjoint approximate identity in the inductive limit topology, $\{e_n\}_{n=1}^{\infty}$. It follows that $\{L(e_n)\}_{n=1}^{\infty}$ converges strongly to the identity operator on H_{00}. More precisely, for $L(f)\eta \in H_{00}$, $||L(e_n)L(f)\eta - L(f)\eta||^2 =$

$$L_{\eta,\eta}(f^* * e_n * e_n * f) - 2\,\mathrm{Re}(L_{\eta,\eta}(f^* * e_n * f)) + L_{\eta,\eta}(f^* * f).$$

Since $\{e_n\}_{n=1}^{\infty}$ is a self-adjoint – and therefore, two-sided – approximate identity for $C_c(G)$ in the inductive limit topology, $e_n * f \to f$, and $f^* * e_n \to f^*$. By the joint continuity of multiplication, $f^* * e_n * e_n * f \to f^* * f$ in the inductive limit topology. Consequently, since $L_{\eta,\eta}$ is a Radon measure, we see that $\lim ||L(e_n)L(f)\eta - L(f)\eta|| = 0$, proving that the span of $L(C_c(G))H_{00}$ is dense in H_0 and, therefore, in H.

For $h \in C_0(G^{(0)})$, we define $M(h)$ on H by first defining $M(h)$ on $C_c(G)\otimes H_0/\mathcal{N}$ via the formula

$$M(h)(\sum f_i \otimes \xi_i + \mathcal{N}) = \sum((h \circ r)f_i) \otimes \xi_i + \mathcal{N}, \tag{3.1}$$

$\sum f_i \otimes \xi_i + \mathcal{N} \in C_c(G) \otimes H_0/\mathcal{N}$. Then $M(h)$ is well defined and bounded by the Effros-Hahn trick [**63**, Page 41]: Write $k(u) = (\|h\|_\infty^2 - |h(u)|^2)^{\frac{1}{2}}$. This is a bounded continuous function on $G^{(0)}$, so $(k \circ r)f \in C_c(G)$ for all $f \in C_c(G)$. We then have $\|M(h)(\sum f_i \otimes \xi_i + \mathcal{N})\|^2 =$

$$\|h\|_\infty^2 \left\|\sum f_i \otimes \xi_i + \mathcal{N}\right\|^2 - \left\|\sum (k \circ r) f_i \otimes \xi_i + \mathcal{N}\right\|^2 \le \|h\|_\infty^2 \left\|\sum f_i \otimes \xi_i + \mathcal{N}\right\|^2.$$

Thus M extends to a C^*- representation of $C_0(G^{(0)})$ on H.

In the usual way (See [**10**].), we may apply reduction theory to produce a measure μ on $G^{(0)}$ and a Hilbert bundle structure, $G^{(0)} * \mathcal{H}$, from H that diagonalizes M. However, for our purposes, it is easier to construct the bundle explicitly, as we hinted above, and to show that the general fibre $\mathcal{H}(u)$ may be taken to be $C_c(G^u) \otimes H_0$ completed with a suitable inner product. After some additional preparation, we shall show that μ is quasi-invariant. We shall then construct the bundle $G^{(0)} * \mathcal{H}$. We shall show that G is represented by left translation on the bundle, and, finally, that L is the integrated form of this representation of G.

We extend M to a C^*-representation of the C^*-algebra of all the bounded Borel functions on $G^{(0)}$ (with the sup-norm) in the standard fashion, keeping the same name M (see [**10**, p.50 ff]). It is worthwhile to emphasize that the extended M is *not* given by formula (3.1) as it stands. Indeed, that equation does not make sense for functions f that are not continuous. We will get around this shortly. The extension annihilates all functions that are null with respect to μ. Thus, really, M may be viewed as a representation of $L^\infty(\mu)$. We also want to extend L to (certain) Borel functions on G, representing them on a dense subspace of H. This is more subtle.

First, we write $B_c(G)$ for the space of bounded functions f on G, such that the support of f is compact and such that there is a uniformly bounded sequence in $C_c(G)$ that is eventually supported in some compact set containing the support of f and that converges to f pointwise through out G. A moment's reflection reveals that $B_c(G)$ is precisely the compactly supported functions in the so-called first Baire class [**78**]. The formulas defining the algebra operations on $C_c(G)$ make sense on $B_c(G)$, giving it the structure of a $*$-algebra. This is not a topological algebra in the usual sense, but there is some structure available that we shall use. First, we shall say that a sequence $\{f_n\}$ in $C_c(G)$ *converges to* f *in* $B_c(G)$ in case $\{f_n\}$ is uniformly bounded, the f_n are supported in a common compact set in G, and $f_n \to f$ pointwise on G. Note that if $\{f_n\}$ converges to f in $B_c(G)$, then by Lebesgue's bounded convergence theorem, $\int f_n \, dm \to \int f \, dm$ for every (Radon) measure m on G. Since the members of a Haar system are Radon measures, we may assert that if $\{f_n\}$ and $\{g_n\}$ converge in $B_c(G)$ to f and g, respectively, then $\{f_n * g_n\}$ converges to $f * g$ in $B_c(G)$. (It is worth mentioning here that $B_c(G)$ is not closed under this kind of convergence: that is, if $\{f_n\}$ is a sequence in $B_c(G)$ that converges pointwise to a function f in such a way that the supports of the f_n are all contained in some prescribed set, then f need not be in $B_c(G)$. In fact, such an f is in Baire class 2. One can iterate this process transfinitely to pick up all compactly supported bounded Borel functions, but we do not know need to go beyond $B_c(G)$.) Our objective is to extend L to $B_c(G)$.

Since the functionals $L_{\xi,\eta}$ are identified with Radon measures, we may evaluate them on $B_c(G)$. We shall show, eventually, that the measures $L_{\xi,\eta}$, with $\xi, \eta \in H_{00}$, are all absolutely continuous with respect to $\nu = \mu \circ \lambda$, but for now we will proceed without this fact[7]. Form the algebraic tensor product $B_c(G) \otimes H_0$ and define $\langle \cdot, \cdot \rangle$ on this space by the same formula we used before: $\langle \sum f_i \otimes \xi_i, \sum g_j \otimes \eta_j \rangle := \sum L_{\xi_i,\eta_j}(f_i^* * g_j)$. This form is positive semi-definite when the functions are in $C_c(G)$ by what was just proved. To prove that it is positive semi-definite on $B_c(G) \otimes H_0$, consider a vector $\sum f_i \otimes \xi_i$ in $B_c(G) \otimes H_0$ and choose a probability measure ω on G with respect to which all the measures L_{ξ_i,ξ_j} are absolutely continuous. Also, for each i choose a sequence $\{f_{i,n}\}_{n=1}^{\infty} \subseteq C_c(G)$ that converges in $B_c(G)$ to f_i. Then, by Lebesgue's bounded convergence theorem, we see that $\langle \sum f_i \otimes \xi_i, \sum f_i \otimes \xi_i \rangle := \sum L_{\xi_i,\xi_j}(f_i^* * f_j)$ $= \lim \sum L_{\xi_i,\xi_j}(f_{i,n}^* * f_{j,n}) = \lim \langle \sum f_{i,n} \otimes \xi_i, \sum f_{i,n} \otimes \xi_i \rangle \geq 0$. Let $\mathcal{N}_b$ be the kernel of the inner product on $B_c(G) \otimes H_0$ and observe that if $\sum f_i \otimes \xi_i \in C_c(G) \otimes H_0 \subseteq B_c(G) \otimes H_0$, then $\sum f_i \otimes \xi_i \in \mathcal{N}$ if and only if $\sum f_i \otimes \xi_i \in \mathcal{N}_b$. Thus the mapping $\sum f_i \otimes \xi_i + \mathcal{N} \to \sum f_i \otimes \xi_i + \mathcal{N}_b$ is isometric and extends to isometry mapping H to the completion of $B_c(G) \otimes H_0/\mathcal{N}_b$, which we shall denote momentarily by H_b. To see that the image of H in H_b is all of H_b, it suffices to show that every vector of the form $g \otimes \xi + \mathcal{N}_b$, with $g \in B_c(G)$ and $\xi \in H_0$, lies in the image of H. But if $\{g_n\}$ is a sequence in $C_c(G)$, converging to g in $B_c(G)$, then by Lebesgue's bounded convergence theorem again,

$$\begin{gathered}
\|g_n \otimes \xi + \mathcal{N}_b - g \otimes \xi + \mathcal{N}_b\|^2 = \\
\|g_n \otimes \xi + \mathcal{N}_b\|^2 - 2\,\mathrm{Re}\langle g_n \otimes \xi + \mathcal{N}_b, g \otimes \xi + \mathcal{N}_b \rangle + \|g \otimes \xi + \mathcal{N}_b\|^2 \\
= L_{\xi,\xi}(g_n^* * g_n) - 2\,\mathrm{Re}\, L_{\xi,\xi}(g_n^* * g) + L_{\xi,\xi}(g * g) \to 0.
\end{gathered}$$

Thus we may view $B_c(G) \otimes H_0/\mathcal{N}_b$ as contained in H so that the following inclusions hold: $H_{00} = C_c(G) \otimes H_0/\mathcal{N} \subseteq B_c(G) \otimes H_0/\mathcal{N}_b \subseteq H$. However, while H_{00} is contained in both $B_c(G) \otimes H_0/\mathcal{N}_b$ and H_0, *a priori* there is no containment relation between $B_c(G) \otimes H_0/\mathcal{N}_b$ and H_0. This is one of the reasons why H_{00} is introduced.

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[7]It will be important for our arguments that ξ and η come from H_{00}. We cannot prove straight away that $L_{\xi,\eta} << \nu$ if either ξ or η is assumed only to lie in H_0. Of course, when the theorem is proved, $L_{\xi,\eta}$ will make sense for all ξ and η in H, and $L_{\xi,\eta} << \nu$. However, it seems to be necessary to work in an incremental fashion.

On H_{00}, $L(f)$, $f \in C_c(G)$, is given by the formula,

$$L(f)(\sum g_i \otimes \xi_i + \mathcal{N}) = \sum (f * g_i) \otimes \xi_i + \mathcal{N},$$

reflecting the homomorphic character of L and the fact that $\mathcal{N} = \{\sum_i f_i \otimes \xi_i \mid \sum L(f_i)\xi_i = 0\}$. Observe that hypothesis (c) of the theorem implies that $L(f^*)$ is contained in the Hilbert space adjoint of $L(f)$, $L(f)^*$. Thus $L(f)^*$ is densely defined, and so $L(f)$ is closable. (See [**169**, Theorem VIII.1].) We write $\overline{L(f)}$ for the closure of $L(f)$. Since the graph of the closure of an operator is the closure of its graph, one checks easily that $B_c(G) \otimes H_0/\mathcal{N}_b$ is contained in $\mathrm{Dom}(\overline{L(f)})$ and that $\overline{L(f)}(\sum g_i \otimes \xi_i + \mathcal{N}_b) = \sum (f * g_i) \otimes \xi_i + \mathcal{N}_b$ for all $\sum g_i \otimes \xi_i + \mathcal{N}_b \in B_c(G) \otimes H_0/\mathcal{N}_b$. (Indeed, if $||\cdot||_\Gamma$ denotes the graph norm of $L(f)$, then given $g \otimes \xi + \mathcal{N}_b$, $g \in C_c(G)$, we have $||g \otimes \xi + \mathcal{N}_b||_\Gamma^2 = ||g \otimes \xi + \mathcal{N}_b||^2 + ||(f * g) \otimes \xi + \mathcal{N}_b||^2 = L_{\xi,\xi}(g^* * g) + L_{\xi,\xi}(g^* * f^* * f * g)$. Since every function g in $B_c(G)$ is the limit of a sequence from $C_c(G)$ converging to g in $B_c(G)$, we conclude that $B_c(G) \otimes H_0/\mathcal{N}_b \subseteq \mathrm{Dom}(\overline{L(f)})$ and the desired formula holds by Lebesgue's bounded convergence theorem.)

Now for $f \in B_c(G)$, define $L_b(f)$ on $B_c(G) \otimes H_0 + \mathcal{N}_b$ by the formula

$$L_b(f)(\sum g_i \otimes \xi_i + \mathcal{N}_b) = \sum f * g_i \otimes \xi_i + \mathcal{N}_b,$$

$\sum g_i \otimes \xi_i + \mathcal{N}_b \in B_c(G) \otimes H_0/\mathcal{N}_b$. We must check that $L_b(f)$ is well defined. If $\sum g_i \otimes \xi_i + \mathcal{N}_b = 0$, but $\sum f * g_i \otimes \xi_i + \mathcal{N}_b \neq 0$, then on the one hand $0 = \overline{L(h)}(\sum g_i \otimes \xi_i + \mathcal{N}_b) = \sum h * g_i \otimes \xi_i + \mathcal{N}_b$ for all $h \in C_c(G)$, while on the other

$$\sum L_{\xi_i,\xi_j}(g_i^* * f^* * f * g_j) > 0.$$

However, this expression may be approximated by expressions of the form

$$\sum L_{\xi_i,\xi_j}(g_i^* * h^* * h * g_j)$$

where $h \in C_c(G)$; and these are all zero since each may be rewritten as $\left\| \overline{L(h)(} \sum g_i \otimes \xi_i + \mathcal{N}_b) \right\|^2$. This contradiction proves that $L_b(f)$, $f \in B_c(G)$, is well defined.

With the definition of $L_b(f)$, $f \in B_c(G)$, in hand, it is straightforward to prove that the map $f \to L_b(f)$ is an algebraic homomorphism of $B_c(G)$ into the (not-necessarily-bounded) linear transformations on $B_c(G) \otimes H_0/\mathcal{N}_b$ with the property that $L_b(f^*) \subseteq L_b(f)^*$, $f \in B_c(G)$. Furthermore, for $f \in C_c(G)$, $L_b(f) = \overline{L(f)}$ on $B_c(G) \otimes H_0/\mathcal{N}_b$.

To lighten the notation, we shall henceforth drop the coset notation for elements in $C_c(G) \otimes H_0/\mathcal{N}$ $(= H_{00})$ and $B_c(G) \otimes H_0/\mathcal{N}_b$; that is, if $g \otimes \xi + \mathcal{N}_b \in B_c(G) \otimes H_0/\mathcal{N}_b$, for example, we shall simply write $g \otimes \xi$. The following lemma highlights the key technical calculations we shall need in the sequel. The reason it is necessary and the thing to keep in mind when using it is that on the one hand, the measures $L_{\xi,\eta}$ are defined in terms of the representation L, its values on functions in $C_c(G)$, and the inner product on H; but on the other hand, L_b and the inner product on elements in $B_c(G) \otimes H_0/\mathcal{N}_b$, are defined in terms of the measures $L_{\xi,\eta}$. It is therefore necessary to make clear the precise roles various constructs are playing.

LEMMA 3.33. *Suppose $f \in B_c(G)$ and that k is a bounded Borel function on $G^{(0)}$ that is the pointwise limit of a bounded sequence in $C_0(G^{(0)})$, i.e., suppose that k is a bounded, Baire class one, function on $G^{(0)}$. Then for all $g \otimes \xi$ and $h \otimes \eta$ in H_{00}, the following equations hold:*

1) $\langle g\otimes\xi, L_b(f)h\otimes\eta\rangle = \langle g\otimes\xi, f*h\otimes\eta\rangle = L_{\xi,\eta}(g^* * f * h) = L_{g\otimes\xi,h\otimes\eta}(f)$.
2) $\langle g\otimes\xi, M(k)h\otimes\eta\rangle = L_{\xi,\eta}(g^* * (k\circ r)h) = \langle g\otimes\xi, (k\circ r)h\otimes\eta\rangle$
$= \langle L(g)\xi, M(k)L(h)\eta\rangle$.
3) $\langle g\otimes\xi, M(k)L_b(f)h\otimes\eta\rangle = \langle g\otimes\xi, L_b((k\circ r)f)h\otimes\eta\rangle$.

PROOF. For 1), observe that the first equation holds simply by definition of $L_b(f)$. The second holds by definition of the inner product on $B_c(G)\otimes H_0/\mathcal{N}_b$. The third holds when $f\in C_c(G)$ by definition of the two measures, $L_{\xi,\eta}$ and $L_{g\otimes\xi,h\otimes\eta}$, and hypothesis c) in the statement of Theorem 3.32. But also, the third equation holds for any f that is the limit of a sequence from $C_c(G)$ that converges to f in $B_c(G)$ by Lebesgue's bounded convergence theorem. The argument for 2) is similar: The first equation holds for all $k\in C_0(G^{(0)})$ by definition of $M(k)$ and the measure $L_{\xi,\eta}$. If $\{k_n\}$ is a bounded sequence in $C_0(G^{(0)})$ converging pointwise to k, then $M(k_n)\to M(k)$ weakly by Lebesgue's bounded convergence theorem. On the other hand, $g^* * (k_n\circ r)h \to g^* * (k\circ r)h$ in $B_c(G)$. Consequently $L_{\xi,\eta}(g^* * (k_n\circ r)h)\to L_{\xi,\eta}(g^* * (k\circ r)h)$ by Lebesgue's bounded convergence theorem again. Therefore the first equation holds. But the second holds for $k\in C_0(G^{(0)})$ by definition of measure $L_{\xi,\eta}$, and in the limit, it holds by definition of the inner product on $B_c(G)\otimes H_0/\mathcal{N}_b$. The third equation holds by virtue of the identification of elements in H_{00} such as $g\otimes\xi$ with vectors of the form $L(g)\xi$. Finally, for 3), observe that 2) implies that $M(k)\varphi\otimes\eta = (k\circ r)\varphi\otimes\eta$ for all $\varphi\otimes\eta$ in $B_c(G)\otimes H_0/\mathcal{N}_b$, since $M(k)$ is bounded and φ is the limit in $B_c(G)$ of a sequence from $C_c(G)$. Also, it is clear that $M(k)^*\varphi\otimes\eta = (\overline{k}\circ r)\varphi\otimes\eta$, for all $\varphi\otimes\eta\in B_c(G)\otimes H_0/\mathcal{N}_b$. Now calculate:

$$\begin{aligned}\langle g\otimes\xi, M(k)L_b(f)(h\otimes\eta)\rangle &= \langle(\overline{k}\circ r)g\otimes\xi, (f*h)\otimes\eta\rangle\\ &= \langle g\otimes\xi, (k\circ r)(f*h)\otimes\eta\rangle\\ &= \langle g\otimes\xi, ((k\circ r)f)*h\otimes\eta\rangle\\ &= \langle g\otimes\xi, L_b((k\circ r)f)(h\otimes\eta)\rangle.\end{aligned}$$

The first equality results from the formulas for $M(k)$ and its adjoint just noted and from the definition of $L_b(f)$. The second, again, is a result of the formulas for $M(k)$. The third is the easy calculation, $(k\circ r)(f*h) = ((k\circ r)f)*h$, which is valid for all functions $f, h\in B_c(G)$ and all bounded Baire class one functions k on $G^{(0)}$. The last equality is by definition of L_b. □

We now prove that the measures $L_{\xi,\eta}$, $\xi,\eta\in H_{00}$, are all absolutely continuous with respect to $\nu = \mu\circ\lambda$. Indeed, suppose f is the characteristic function of a Borel set that is null for ν. By the regularity of ν, we may assume without loss of generality that the null set is a G_δ. That is, we may assume that f lies in $B_c(G)$. From the equation

$$0 = \int f\,d\nu = \int_{G^{(0)}}\int f(x)\,d\lambda^u(x)\,d\mu(u),$$

we infer that there is a μ-null set N in $G^{(0)}$ such that $f(x)=0$, a.e. λ^u for all $u\notin N$. Again, by regularity – of μ this time – N is contained in a null G_δ, and so we may assume without loss of generality that N itself is a null G_δ. In particular, we may assume that 1_N is the bounded, pointwise limit of a sequence of functions from $C_0(G^{(0)})$. Since $f(x)=0$ a.e. λ^u for all $u\notin N$, we conclude that for each $h\in B_c(G)$,

$$f*h(x) = 1_N(r(x))f*h(x) \tag{3.2}$$

for all $x \in G$ *without exception.* Suppose that we are given vectors in H_{00}. We may assume, without loss of generality, that they are of the form $g \otimes \xi$ and $h \otimes \eta$, with $g, h \in C_c(G)$, and $\xi, \eta \in H_0$. Since f and 1_N satisfy the hypotheses of Lemma 3.33, we may appeal to assertion 1) in it to write $L_{g\otimes\xi,h\otimes\eta}(f) = \langle g \otimes \xi, f * h \otimes \eta\rangle$. By equation (3.2), this may be written as $\langle g \otimes \xi, ((1_N \circ r)f) * h \otimes \eta\rangle$. By assertions 1) and 3) of Lemma 3.33, this, in turn, may be written as $\langle g \otimes \xi, M(1_N)L_b(f)h \otimes \eta\rangle$. However, since N is a null set for μ, $M(1_N) = 0$. Thus all the expressions are zero; in particular, $L_{g\otimes\xi,h\otimes\eta}(f) = 0$, as we wanted to prove. Thus $L_{g\otimes\xi,h\otimes\eta} << \nu$.

We now show that μ is quasi-invariant, i.e., that ν is equivalent to ν^{-1}. Let h be the characteristic function of a null set. We shall show that $\tilde{h} = 0$ a.e. ν, where $\tilde{h}(x) = h(x^{-1})$. To this end, we may assume, by the regularity of ν that the null set is, in fact, a G_δ, so that $h \in B_c(G)$. For every $k \in C_c(G)$, the function $|k|^2 h$ (pointwise product) is a function in $B_c(G)$ that vanishes a.e. ν. So if we set $\lambda(\overline{k}hk) = \int |k|^2 h\, d\lambda^u$, then $\lambda(\overline{k}hk)$ is a bounded, Baire class one function on $G^{(0)}$ by the monotone convergence theorem, and $\lambda(\overline{k}hk) = 0$, a.e. μ. Applying part 2) of Lemma 3.33, we may write

$$L_{\xi,\xi}(g^* * (\lambda(\overline{k}hk) \circ r)g) = \langle L(g)\xi, M(\lambda(\overline{k}hk))L(g)\xi\rangle = 0 \tag{3.3}$$

for all $g \in C_c(G)$ and $\xi \in H_{00}$. The converse also holds, i.e., if this equation yields zero for all g, ξ, and k, then $h = 0$, a.e. ν. Indeed, if the equation yields zero for all $g \in C_c(G)$ and $\xi \in H_{00}$, then the operator $M(\lambda(\overline{k}hk)) = 0$, since as we showed when we introduced H_{00}, the span of $L(C_c(G))H_{00}$ is dense in H. But if $M(\lambda(\overline{k}hk)) = 0$, then $\lambda(\overline{k}hk) = 0$ a.e. μ, and so $\overline{k}hk = 0$ a.e. ν. If this happens for every $k \in C_c(G)$, we conclude that $h = 0$ a.e. ν. Thus, it suffices to show

$$L_{\xi,\xi}(g^* * (\lambda(\overline{k}\tilde{h}k) \circ r)g) = 0 \tag{3.4}$$

for all g, $k \in C_c(G)$, and $\xi \in H_{00}$. A calculation shows that

$$g^* * (\lambda(\overline{k}hk) \circ r)g = f^* *_{\lambda*\lambda} (hf)$$

where f and hf are defined on $G *_r G := \{(x, y) \in G \times G \mid r(x) = r(y)\}$ by the formulas $f(x, y) = k(y^{-1}x)g(y)$ and $hf(x, y) = h(x^{-1}y)f(x, y)$, and where

$$f_1^* *_{\lambda*\lambda} f_2(z) := \int\int \overline{f_1}(zx, zy)f_2(x, y)\, d\lambda^{s(z)}(x)\, d\lambda^{s(z)}(y),$$

for f_1 and f_2 in $B_c(G *_r G)$. Note that our particular f lies in $C_c(G *_r G)$ and linear combinations of functions of the same form as f, obtained by letting g and k range over $C_c(G)$, are dense in $C_c(G *_r G)$ by the Stone-Weierstrass theorem. Hence for *every* $f \in C_c(G *_r G)$, $f^* *_{\lambda*\lambda} (hf)$ is a compactly supported bounded Borel function on G so that

$$L_{\xi,\xi}(f^* *_{\lambda*\lambda} (hf)) = 0 \tag{3.5}$$

for all $\xi \in H_{00}$, by equation (3.3). Now observe that $\lambda * \lambda$ is invariant under the flip $(x, y) \to (y, x)$. Consequently,

$$f^* *_{\lambda*\lambda} (hf) = \tilde{f}^* *_{\lambda*\lambda} (\tilde{h}\tilde{f})$$

for all $f \in C_c(G * G)$, where $\tilde{f}(x, y) = f(y, x)$ and, recall, $\tilde{h}(x) = h(x^{-1})$. Thus equation (3.5) implies that $L_{\xi,\xi}(f^* *_{\lambda*\lambda} (\tilde{h}f)) = 0$ for all $f \in C_c(G*G)$ and $\xi \in H_0$. This, in turn, yields equation (3.4) and completes the proof that μ is quasi-invariant.

Set $\nu_0 = \Delta^{-\frac{1}{2}}\nu$, where, recall, $\nu = \mu \circ \lambda$ and $\Delta = d\nu/d\nu^{-1}$. Recall, too, that by Theorem 3.15, Δ may be chosen to be a homomorphism from G into $\mathbb{R}_+^\times$. Also, for ξ and η in H_{00}, let $\rho_{\xi,\eta}$ be the Radon-Nikodym derivative of $L_{\xi,\eta}$ with respect to ν_0. Then we may write

$$\langle \xi, L(f)\eta \rangle = \int f(x)\rho_{\xi,\eta}(x)\, d\nu_0$$

for all $f \in C_c(G)$. For each $u \in G^{(0)}$ we define $\langle \cdot, \cdot \rangle_u$ on $C_c(G) \otimes H_{00}$ by the formula

$$\langle f \otimes \xi, g \otimes \eta \rangle_u = \int \int \overline{f(x)} g(y) \rho_{\xi,\eta}(x^{-1}y) \Delta^{-\frac{1}{2}}(x) \Delta^{-\frac{1}{2}}(y)\, d\lambda^u(x)\, d\lambda^u(y).$$

Observe, first, that the value of $\langle f \otimes \xi, g \otimes \eta \rangle_u$ only depends on the values of f and g restricted to G^u. Then observe that Fubini's theorem implies that for all $h \in C_c(G^{(0)})$ and all $f,\, g \in C_c(G)$

$$\int h(u) \langle f \otimes \xi, g \otimes \eta \rangle_u \, d\mu(u) = \langle M(h)L(f)\xi, L(g)\eta \rangle \tag{3.6}$$

for all $\xi,\, \eta \in H_{00}$, and that for all $f\,, g \in C_c(G)$ and for all $\xi,\, \eta \in H_{00}$,

$$\langle x \cdot f \otimes \xi, x \cdot g \otimes \eta \rangle_{r(x)} = \Delta^{-1}(x) \langle f \otimes \xi, g \otimes \eta \rangle_{s(x)}, \tag{3.7}$$

for ν-almost all $x \in G$, where $(x \cdot f)$ denotes any function in $C_c(G)$ such that $(x \cdot f)(y) = f(x^{-1}y)$, $y \in G^{r(x)}$. (Such functions exist by the Tietze extension theorem.) It should be emphasized that the exceptional ν-null set in (3.7) depends upon f, g, ξ, and η. We will handle such null sets by focusing on a suitable countable collection of functions.

To this end, choose an orthonormal basis $\{\xi_i\}$ for H_{00} and a sequence $\{\varphi_i\}$ in $C_c(G)$ that is dense in the inductive limit topology. By equation (3.6) there is a μ-conull set $F \subseteq G^{(0)}$ such that for $u \in F$, $\langle \cdot, \cdot \rangle_u$ is a positive semi-definite sesquilinear form on the vector space (over $\mathbb{Q} + \sqrt{-1}\mathbb{Q}$) generated by $\{\varphi_i \otimes \xi_j\}$. By continuity of the integral, positivity is preserved on the $\mathbb{C}$-vector space generated by $\{f \otimes \xi_i |\, f \in C_c(G)\}$. For $u \in F$, let $\mathcal{H}(u)$ be the Hausdorff completion of this vector space with respect to $\langle \cdot, \cdot \rangle_u$, and write $f \otimes_u \xi$ for the image of $f \otimes \xi$ in $\mathcal{H}(u)$. If we define sections $\Phi_{ij} : F \to F * \mathcal{H}$ by the formula $\hat{\Phi}_{ij}(u) = \varphi_i \otimes_u \xi_j$, then by Proposition 3.2, $F * \mathcal{H}$ becomes a Hilbert bundle over F with $\{\Phi_{ij}\}$ as a fundamental sequence. Sweeping all the exceptional ν-null sets associated with the sequences $\{\varphi_i\}$ and $\{\xi_j\}$ in equation (3.7) into one grand null set and taking its complement V, we may conclude from equation (3.7) that for all $x \in V \cap (G|_F)$, the formula

$$\hat{U}(x)(f \otimes_{s(x)} \xi) = \Delta^{\frac{1}{2}}(x)(x \cdot f \otimes_{r(x)} \xi) \tag{3.8}$$

defines an isometry from $\mathcal{H}(s(x))$ onto $\mathcal{H}(r(x))$. Observe that the set of $x \in G|_F$ such that (the right hand side of) equation (3.8) defines an isometry from $\mathcal{H}(s(x))$ onto $\mathcal{H}(r(x))$ is closed under multiplication. Since this set contains the ν-conull set $V \cap (G|_F)$, Lemma 3.21 implies that there is a μ-conull set F' such that $G|_{F'} \subseteq V \cap (G|_F)$. Replacing F by F', if necessary, we obtain a homomorphism

$$U : G|_F \to \mathrm{Iso}(F * \mathcal{H}).$$

This map is clearly Borel by virtue of the equation $\langle f \otimes_{r(x)} \xi, \hat{U}(x)(g \otimes_{s(x)} \eta) \rangle_{r(x)} =$

$$\int \int f(y_1) \overline{g(x^{-1}y_2)} \rho_{\xi,\eta}(y_1^{-1}y_2) \Delta^{-\frac{1}{2}}(y_1) \Delta^{-\frac{1}{2}}(y_2) \Delta^{\frac{1}{2}}(x)\, d\lambda^{r(x)}(y_1)\, d\lambda^{r(x)}(y_2).$$

Since $G|_F$ is conull in G with respect to the measure ν, we may apply Theorem 3.20 to conclude that there is a Borel homomorphism of G into $\mathrm{Iso}(G^{(0)} * \mathcal{H})$, which we shall continue to denote by U, that satisfies equation (3.8) on a ν-conull subset of G.

If the Φ_{ij} are viewed as elements in the direct integral, $\int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\,d\mu(u)$ and if we write $V(L(\varphi_i)\xi_j) = \Phi_{ij}$, then

$$\begin{aligned}\langle V(L(\varphi_i)\xi_j), V(L(\varphi_k)\xi_l)\rangle &= \int \langle \varphi_i \otimes_u \xi_j, \varphi_k \otimes_u \xi_l\rangle_u \, d\mu(u) \\ &= \langle L(\varphi_i)\xi_j, L(\varphi_k)\xi_l\rangle\end{aligned}$$

and so V extends to a Hilbert space isomorphism from H onto $\int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\,d\mu(u)$. (Note that V extends to all of H since $\{\xi_j\}$ is an orthonormal basis for H_{00} and $\{\varphi_i\}$ is a dense subset of $C_c(G)$.) It is then a straightforward computation to check that (on a dense subset of $\int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\,d\mu(u)$) $VL(f)V^{-1} = L'(f)$, $f \in C_c(G)$, where L' is the integrated form of U. More precisely, we see that for vectors of the form $L(\varphi_i)\xi_j$ we have

$$\begin{aligned}&\langle L(f)L(\varphi_i)\xi_j, L(\varphi_k)\xi_l\rangle = \langle VL(f)L(\varphi_i)\xi_j, VL(\varphi_k)\xi_l\rangle \\ &= \langle L'(f)VL(\varphi_i)\xi_j, VL(\varphi_k)\xi_l\rangle = \int f(x)\langle \hat{U}(x)(\varphi_i \otimes_{s(x)} \xi_j), (\varphi_k \otimes_{r(x)} \xi_l)\rangle_{r(x)}\, d\nu_0(x).\end{aligned}$$

This completes the proof. □

Theorem 2.42 is now an immediate corollary of Theorem 3.32 and Proposition 3.25. It is worthwhile to emphasize that we usually apply Theorem 3.32 to C^*-representations of $C^*(G,\lambda)$. This is certainly permissible. For if a representation π of $C^*(G,\lambda)$ is given, then its restriction to $C_c(G)$ clearly satisfies the hypotheses of Theorem 3.32. Therefore, we may disintegrate π in terms of a uniquely determined representation of G, $(\mu, G^{(0)} * \mathcal{H}, L)$, and write $\pi(f)\xi(u) = \int f(x)(U(x)\xi(s(x)))\Delta^{-\frac{1}{2}}(x)\,d\lambda^u(x)$, for all $\xi \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\,d\mu(u)$.

It may seem that the hypotheses of Theorem 3.32 are a little contrived. For example, one may wonder why worry about representations of $C_c(G)$ that act on a linear submanifold of a Hilbert space. It turns out that this provides a degree of freedom in arguments that is very useful. Indeed, as we shall see in Chapter 5 when we discuss Morita equivalence for groupoids the formulation of Theorem 3.32 that we gave will be crucial. See the proof of Theorem 5.38 in particular.

The following corollary should be compared with Proposition 2.47.

COROLLARY 3.34. *If $G = X \times X$ is the trivial groupoid determined by a second countable, locally compact, Hausdorff space, X, and if the Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ is given by the formula $\lambda^u = \epsilon_u \times \lambda$ where λ is a Radon measure on X with* $\mathrm{supp}(\lambda) = X$, *then $C^*(G,\lambda)$ is an elementary C^*-algebra.*

PROOF. Let π be a representation of $C^*(G,\lambda)$, and let $(\mu, G^{(0)} * \mathcal{H}, L)$ be its disintegration. We know from Exercise 3.17 that μ and λ are mutually absolutely continuous. We may therefore replace μ by λ in all that follows. When this is done, we see that $\nu = \lambda \times \lambda$ is obviously invariant, i.e., $\Delta \equiv 1$. The bundle $G^{(0)} * \mathcal{H}$ is isomorphic to a constant bundle $G^{(0)} \times H$. Indeed, observe that in the representation of any groupoid, the sets where $\dim(\mathcal{H}(u))$ is constant are invariant. Since there is only one, nonempty, invariant set in this example, namely X, the assertion follows from Proposition 3.11. By the remarks after Definition 3.26, we may write the Hilbert space of π as $L^2(\lambda, H)$ and for ξ in it, and $f \in C_c(G)$, we may write $(\pi(f)\xi)(x) = \int f(x,y)(L_0(x,y)\xi)(y)\,d\lambda(y)$, where $L_0 : G \to \mathcal{U}(H)$, is the

Borel homomorphism determined by L. However, we may apply Proposition 3.19 to conclude that there is a Borel function $B : X \to \mathcal{U}(H)$ such that $L_0(x, y) = B(x)B(y)^{-1}$, for all x, $y \in X$. If we define $W : L^2(\lambda, H) \to L^2(\lambda, H)$ by the formula $(W\xi)(x) = B(x)^{-1}\xi(x)$, then

$$(W\pi(f)W^{-1}\xi)(x) = \int f(x, y)\xi(y)\, d\lambda(y).$$

Thus π, restricted to $C_c(G)$, is unitarily equivalent to a multiple of the canonical representation of $C_c(G)$ on $L^2(\lambda)$. This shows that $C^*(G, \lambda)$ is isomorphic to the algebra of compact operators on $L^2(\lambda)$. $\square$

CHAPTER 4

Measured Groupoids

In this chapter, we concentrate on some of the measure theoretic aspects of groupoids. Our primary objective is to give complete proofs of Theorem 3.20 and Lemma 3.21. These played key roles in the proof of Renault's disintegration theorem, Theorem 3.32. This we do in the next two sections. The remainder of the chapter surveys some of the theory of measured groupoids. In the third section, we present a brief discussion of Peter Hahn's work on Haar measure for measured groupoids and the von Neumann algebra of a measured groupoid. Section 4 is devoted to the theory of transversals for measured groupoids. Section 5 is concerned with the mapping properties of measured groupoids.

Revise, taking into account the introduction of Borel systems of measures in Chapter3.

Add a section on Connes's transvers measuress? Mention comparisons between similar groupoids and equivalent groupoids done in Chapter 5?

1. Preliminaries

In this chapter, we rescind our blanket assumption that all groupoids are locally compact, Hausdorff, and second countable. We will invoke it as needed. Instead, throughout this chapter, unless otherwise asserted, G will denote an analytic Borel groupoid. Recall from Definition 2.25 that this means that G has a Borel structure, which we are assuming is analytic, that $G^{(2)}$ is a Borel subset of $G \times G$, and that the groupoid operations are Borel maps. Recall, also, that a measure class C on a Borel space X is a family of (σ-finite) measures on X that are mutually absolutely continuous. We assume that measure classes are complete in the sense that if the class contains one measure, μ say, then it contains every other measure that is mutually absolutely continuous with respect to μ. Sometimes it is useful to single out a measure μ, say, and the measure class it determines. In such events, we write $[\mu]$ for the class μ determines.[1] In Chapter 1, we discussed invariant measure classes on spaces on which groups act. Here, we need to generalize this notion to groupoids.

Definition 4.1. *Let G be an analytic Borel groupoid and let C be a measure class on G.*

1. *The class C is called* symmetric *if and only if ν is mutually absolutely continuous with ν^{-1}, where, recall, ν^{-1} is the image of ν under inversion ($x \to x^{-1}$).*
2. *Let $\nu \in C$ be a probability measure and let $\nu = \int \nu^u \, d\mu(u)$ be its r-decomposition (See Definition 3.7.) Suppose there is a μ-conull set $U \subseteq G^{(0)}$ so that*

[1] This notation conflicts with our earlier use of $[\mu]$ to denote the saturation of a measure μ. However, since we will not be considering measure classes and saturations simultaneously, the dual use of $[\mu]$ should not cause confusion.

for $x \in G|_U$, $x \cdot \nu^{s(x)} \sim \nu^{r(x)}$.[2] *Then* ν *is said to have a* quasi-invariant r-decomposition *and* $\{\nu^u\}_{u\in G^{(0)}}$ *is called the* quasi-invariant family associated with ν.

3. *If* C *is symmetric and contains a probability measure with quasi-invariant* r*-decomposition, then* C *is called* invariant.
4. *If* C *is invariant, the pair* (G, C) *is called a* measured groupoid.

REMARK 4.2. *In some places in the literature, a probability measure on* G *with a quasi-invariant* r*-decomposition is called quasi-invariant. However, here, we have reserved the term 'quasi-invariant' to refer to measures on the unit space satisfying the condition of Definition 3.14.*

It is easy to see that if C is a symmetric measure class, then C contains probability measures ν that are *symmetric* in the sense that $\nu = \nu^{-1}$: If $\nu_0 \in C$, then so is $\nu := \frac{1}{2}(\nu_0 + \nu_0^{-1})$, and ν is symmetric. It is also easy to see, using Theorem 3.6, that if C is an invariant measure class, then every probability measure in C has a quasi-invariant r-decomposition as well as a quasi-invariant s-decomposition (which is defined in the obvious way).

Examples of measured groupoids are easy to come by: Simply let G be a locally compact groupoid with a Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, select a quasi-invariant measure μ on $G^{(0)}$ (such measures exist by Remark 3.18) and let C be the measure class determined by $\int \lambda^u \, d\mu(u)$. The question immediately arises: Is this the only way to get measured groupoids? That is, if (G, C) is a measured groupoid, must G have a locally compact topology and must C come from a Haar system? We shall have more to say about this a little later.

Contact Arlan later on the status of his theorem.

A special role is played by the trivial groupoid $G = [0, 1]^2$ and the invariant measure class C determined by area measure. We call this the *trivial uncountable measured groupoid.*

Suppose that (G, C) is a measured groupoid and that ν is a probability measure in C. Then we can form the probability measure $r(\nu)$ on $G^{(0)}$ and the measure class that $r(\nu)$ determines. This measure class is easily seen to be independent of the choice of $\nu \in C$, and so we denote it $r(C)$.

DEFINITION 4.3. *A measured groupoid* (G, C) *is called an* ergodic *measured groupoid or a* virtual group *in case the measure class* $r(C)$ *is ergodic in the sense that every invariant Borel subset of* $G^{(0)}$ *is either null or conull for* $r(C)$.

This definition of "virtual group" is Ramsay's [**157**, p. 274]. Mackey defines a virtual group to be a *similarity class* of ergodic measured groupoids. See [**120, 121, 123**] and the discussion below.

One proves without too much difficulty that if G is a transformation group groupoid $X \times H$ determined by a locally compact group H acting on a locally compact space X and if C is the measure class determined by the measure $\mu \times \lambda$, where λ is Haar measure on H and μ is a quasi-invariant measure on X, then C is invariant and (G, C) is ergodic if and only if μ is ergodic in the usual sense. See [**157**, Theorem 4.3] for details.

It is useful to note that there is a slightly different way of expressing ergodicity that is a bit easier to deal with in practice. Suppose (G, C) is a measured

[2] Recall that $x \cdot \nu^{s(x)}$ denotes the image of $\nu^{s(x)}$ under the transformation that x implements from $G^{s(x)}$ to $G^{r(x)}$. Thus $\int f(y) \, d(x \cdot \nu^{s(x)})(y) := \int f(xy) \, d\nu^{s(x)}(y)$.

groupoid and let $A \subseteq G^{(0)}$ be a Borel subset. Then A is called *almost invariant* iff $s^{-1}(A)\Delta r^{-1}(A)$ is a null set for C. (Here, Δ denotes symmetric difference.) In [**157**, Theorem 4.2], Ramsay proves that C is ergodic iff every almost invariant set is null or conull for $r(C)$. In turn, this happens iff each Borel function f on $G^{(0)}$ such that $f \circ r = f \circ s$ a.e.C, necessarily is constant a.e.$r(C)$. In the case of a transformation group $X \times H$, just discussed, a subset $A \subseteq X$ is almost invariant iff for each $t \in H$, $At = A$ a.e.μ. Thus, in this case, it is easy to see that almost invariant sets differ from invariant sets by null sets.

Rewrite this. Perhaps put in some of Mackey's analysis.

As we indicated in Chapter 1, Mackey realized that virtual groups are a natural generalization of transitive group actions. A bit more precisely, if H is a subgroup of a group G one can express properties of H in terms of the transformation group $G \times G/H$. For example, one can express a homomorphism from one group H_1, contained, say, in G_1, to H_2 contained in G_2 in terms of a groupoid homomorphism from $G_1 \times G_1/H_1$ to $G_2 \times G_2/H_2$. If H_1 and H_2 are subgroups of the same group G, then it is possible to express whether or not H_1 and H_2 are conjugate in terms of the *similarity* of the groupoids $G \times G/H_1$ and $G \times G/H_2$. Once the transition from H to $G \times G/H$ is made, it is natural pass from $G \times G/H$ to general ergodic groupoid and a homomorphism of it into G, i.e. to a virtual subgroup of G. One is faced with a number of problems relating to the nature of homomorphisms between measured groupoids in carrying out this transition smoothly. These were taken up by Ramsay [**157**] in a systematic way and we turn to part of his analysis now.

2. Homomorphisms of measured groupoids. I

When working with a measured groupoid, it is often necessary to pare it down slightly to a smaller groupoid called an inessential contraction or inessential reduction. We have met this term before in connection with representations. We repeat the definition here to capture a point that will be useful to our discussion. Suppose (G, C) is a measured groupoid and $U \subseteq G^{(0)}$ is a subset of $G^{(0)}$ that is conull with respect to $r(C)$. Then because C is an invariant measure class, $G|_U = r^{-1}(U) \cap s^{-1}(U)$ is a conull subset of G. If one denotes the restriction of the measures in C to $G|_U$ by $C|_U$, then $(G|_U, C|_U)$ is a measured groupoid that is called the *inessential contraction of G to U*.

If (G, C) is a measured groupoid, and if $\nu \in C$ is a *symmetric*, $\nu = \nu^{-1}$, then $r(\nu) = s(\nu)$ and if we denote the common value by μ, we will write $\nu = \int \nu^u \, d\mu(u)$ for the r-decomposition of ν and $\nu = \int \nu_u \, d\mu(u)$ for the s-decomposition of ν. We then have: For u in a μ-conull subset of $G^{(0)}$, $\nu^u(E) = \nu_u(E^{-1})$ for every Borel set E in G. That is, the exceptional null set of u is independent of E. We may assume, then, in the r- and s- decompositions of ν, that $\nu^u(E) = \nu_u(E^{-1})$ for all $u \in G^{(0)}$ and all Borel sets E without exception. The measure families, $\{\nu_u\}_{u \in G^{(0)}}$ and $\{\nu^u\}_{u \in G^{(0)}}$, give rise to a measure on $G^{(2)}$, denoted $\nu^{(2)}$, and defined by the formula $\nu^{(2)} = \int \nu_u \times \nu^u \, d\mu(u)$. The measure class on $G^{(2)}$ determined by $\nu^{(2)}$ is denoted $C^{(2)}$.

EXAMPLE 4.4. *The following discussion may be useful to illustrate some of the ideas we have been discussing. It will play a role in the next section. Suppose that (G, C) is a measured groupoid. Recall from Remarks 2.14 that $G^{(2)}$ is a groupoid in its own right; it is the action groupoid determined by G acting on G on the right. The fact that G is analytic (standard) implies that $G^{(2)}$ is also analytic (standard).*

This is not difficult to show [**87**, Lemma 3.3]. *It is only a little harder to show that* $C^{(2)}$ *is an invariant measure class on* $G^{(2)}$. *See* [**87**, Proposition 3.4] *for details.*

REMARK 4.5. *The measured groupoid* $(G^{(2)}, C^{(2)})$ *is ergodic if and only if* C *is concentrated on an orbit. To see this, first observe that* $r^{(2)}(\nu^{(2)}) = \nu$ *and that the* $r^{(2)}$*-decomposition of* $\nu^{(2)}$ *is* $\nu^{(2)} = \int \delta_x \times \nu^{s(x)}\, d\nu(x)$, *i.e.,*

$$\int f(x,y)\, d\nu^{(2)} = \int\int\int f(x,y)\, d\nu^{s(x)}(y)\, d\nu^u(x)\, d\mu(u).$$

Second, observe that if ϕ *is any Borel function on* $G^{(0)}$, *then setting* $g(x, s(x)) = \phi \circ r(x)$ *defines a Borel function* g *on* $(G^{(2)})^{(0)}$ *that is invariant:* $g \circ r^{(2)} = g \circ s^{(2)}$. *Now, if* $C^{(2)}$ *is ergodic, then every such invariant function* g *must be a.e.constant with respect to* $r^{(2)}(\nu^{(2)}) = \nu$. *This happens iff* ϕ *is constant a.e.*μ. *But if every Borel function on* $G^{(0)}$ *is constant a.e.*μ, μ *must be a point mass. Conversely, if* μ *is a point mass, then it follows from the* $r^{(2)}$*-decomposition of* $\nu^{(2)}$, *that* $\nu^{(2)}$ *is concentrated on the transitive groupoid* $G^{(2)}|_{G^u}$ *and so is ergodic.*

The following terminology is adopted from [**161**]. It differs slightly from earlier terminology in [**157**] and elsewhere in the literature.

DEFINITION 4.6. *Let* G *and* H *be analytic Borel groupoids and let* $\varphi : G \to H$ *be a Borel map.*

1. φ *is called a* homomorphism *or a* Borel homomorphism *if algebraically* φ *is a homomorphism in the sense of Definition 2.4.*
2. *If* C *is an invariant measure class on* G, *so that* (G, C) *is a measured groupoid, then* φ *is called a* weak homomorphism *(with respect to* C*) in case there is an inessential contraction of* G, $G|_U$, *so that the restriction of* φ *to* $G|_U$ *is a homomorphism in the sense just defined.*
3. *If* C *is an invariant measure class on* G, *the* φ *is called an* a.e.homomorphism *(with respect to* C*) if the set* $\{(x,y) \in G^{(2)} \mid \varphi(x)\varphi(y) = \varphi(xy)\}$ *is conull with respect to* $C^{(2)}$.

It may be helpful to note that weak homomorphisms sometimes receive casual treatment. If ϕ is a homomorphism of an inessential contraction $G|_U$ of G, then one can treat ϕ as a weak homomorphism on G simply by defining ϕ to be an arbitrary constant value on the complement of $G|_U$. Also, of course, a weak homomorphism may have many inessential reductions on which it is a homomorphism. These sorts of ambiguities should cause no difficulties in our discussion.

REMARK 4.7. *With Definition 4.6 in hand, it is worthwhile to continue Remark 4.4 and to note that* $G^{(2)}$ *is cohomologically trivial in the sense that if* $\sigma : G^{(2)} \to H$ *is a (algebraic) homomorphism, then there is a function* $b : G \to H$ *such that* $\sigma(x,y) = b(r^{(2)}(x,y))b(s^{(2)}(x,y))^{-1}$ *for all* $(x,y) \in G^{(2)}$. *The function* b *may be chosen to be Borel if and only if* σ *is Borel. Indeed, simply set* $b(x) = \sigma(x, x^{-1})$ *and recall that* $r^{(2)}(x,y) = (x, s(x))$ *which we identify with* x, *while* $s^{(2)}(x,y) = (xy, s(xy)) = (xy, s(y))$ *which we identify with* xy. *A straightforward calculation shows that* $\sigma(x,y) = b(r^{(2)}(x,y))b(s^{(2)}(x,y))^{-1}$.

The proofs of the following two results are taken directly from Ramsay's article [**157**]. He attributes a key element of the proof of the following theorem to Lemma 6.2 of [**118**]. Note that in [**157**], the hypotheses include the assumption that the invariant measure classes in question are ergodic. However, the proofs don't use ergodicity.

THEOREM 4.8. [**157**, Theorem 5.2] *Let (G, C) be a measured groupoid and suppose $\varphi : G \to H$ is an a.e.homomorphism, where H is an analytic Borel groupoid. Then there is a weak homomorphism $\varphi_0 : G \to H$ such that $\varphi_0 = \varphi$ a.e. C.*

This result rests on the following lemma which, when specialized to a measured groupoid coming from a locally compact groupoid with a Haar system and a quasi-invariant measure, is Lemma 3.21. It is an analogue of the well known fact that a conull subsemigroup of a locally compact group is the entire group.

LEMMA 4.9. [**157**, Lemma 5.2] *Let (G, C) be a measured groupoid, let Σ be a subset of G that is closed under multiplication and assume that Σ contains a conull set for C. Then there is a set $U \subseteq G^{(0)}$ such that $G|_U$ is contained in Σ and U is conull with respect to $r(C)$.*

PROOF. Set $\Sigma_1 = \Sigma \cap \Sigma^{-1} = \{x \in G | x, x^{-1} \in \Sigma\}$. Then Σ_1 is a groupoid contained in G (whose unit space may be smaller than $G^{(0)}$) and contains a conull Borel subset B of G. Let ν be a probability measure in C and write its s-decomposition as $\nu = \int \nu_u d\mu(u)$. Then μ is a probability measure on $G^{(0)}$ and μ-almost every ν_u is a probability measure on G. (The support of ν_u is contained in G_u, recall.) The invariance assumption on C means that $\nu_{r(x)} \cdot x \sim \nu_{s(x)}$ for ν-almost all x. Since B is conull with respect to C and therefore with respect to ν, we may find a μ-conull set $U \subseteq G^{(0)}$ such that for all $u \in U$, $\nu_u(G) = \nu_u(B) = \nu_u(G_u \cap B) = 1$, and such that for all x satisfying $r(x)$, $s(x) \in U$ (i.e., for all $x \in G|_U$), $\nu_{r(x)} \cdot x$ and $\nu_{s(x)}$ are equivalent. We want to show that $G|_U$ is contained in Σ_1. Let $x \in G|_U$ and write $u = s(x)$ and $v = r(x)$. So u and v lie in U. By assumption on U, we have $\nu_v(G_v \cap B) = 1$, so $G_v \cap B$ is conull with respect to ν_v, and $\nu_u \cdot x \sim \nu_v$. Therefore, $(G_v \cap B)x$ is conull with respect to ν_u. On the other hand, since $u \in U$, $G_u \cap B$ is also conull with respect to ν_u. Consequently, $(G_v \cap B)x \cap G_u \cap B$ is conull with respect to ν_u. In particular, this set is not empty. Thus, there are y and z in $B \subseteq \Sigma_1$ such that $y = zx$. Therefore $x = z^{-1}y \in \Sigma_1$. □

PROOF OF THEOREM 4.8.

Fix a symmetric probability measure ν in C and write its r-decomposition as $\nu = \int \nu^u \, d\mu(u)$. Since ν is symmetric, it has s-decomposition $\nu = \int \nu_u \, d\mu(u)$ with $\nu^u(E) = \nu_u(E^{-1})$ for every Borel set E in G. The measure class $C^{(2)}$ is determined by $\nu^{(2)} = \int \nu_u \times \nu^u \, d\mu(u)$. By hypothesis, there is a μ-conull set $U_1 \subseteq G^{(0)}$ such that for $u \in U_1$, $\varphi(x)\varphi(y)$ is defined and equal to $\varphi(xy)$ for $\nu_u \times \nu^u$-almost all pairs (x, y) in $G_u \times G^u$. As in the proof of the preceding lemma, we may assume without loss of generality that $\nu_u(G_u) = 1$ for all $u \in U_1$ and that $\nu_{r(x)} \cdot x \sim \nu_{s(x)}$ for all $x \in G_1 := G|_{U_1}$. Since H is assumed to be analytic, H is Borel isomorphic to an analytic subset of the unit interval $[0, 1]$. We shall think of H as sitting in this interval via such an isomorphism. Also, fix an element $h \in H$ arbitrarily. We define a function $f : G \times G \to H$ by the formula:

$$f(x, y) = \begin{cases} \varphi(x)^{-1}\varphi(xy), & \text{if the product is defined,} \\ h, & \text{otherwise.} \end{cases}$$

Since the set of (x, y) for which $\varphi(x)^{-1}\varphi(xy)$ is defined is a Borel set and φ is Borel, f is a Borel function, which we think of as having values in the unit interval. Define

$\varphi_1 : G \to [0,1]$ by the formula $\varphi_1(y) = \int f(x,y)\, d\nu_{r(y)}(x)$. Then φ_1 is a Borel function of y. (Indeed, the set of non-negative functions g such that $\int g(x,y)\, d\nu_{r(y)}(x)$ is Borel function is a cone that is closed under monotone limits and contains all product functions, $g(x,y) = h(x)k(y)$. So it contains all non-negative Borel functions on $G \times G$.) Further, for ν-almost all y, the set of $x \in s^{-1}(r(y))$ such that $\varphi(x)\varphi(y)$ is defined and equal to $\varphi(xy)$ is conull. Consequently, $\varphi_1 = \varphi$, a.e. ν.

Now let $G_2 = \{y \in G | \varphi(x)^{-1}\varphi(xy)$ is defined and constant a.e. $\nu_{r(y)}$ on $s^{-1}(r(y))\}$. Observe that G_2 contains the set $B = \{y | \varphi(xy) = \varphi(x)\varphi(y)$ for $\nu_{r(y)}$-almost all $x \in s^{-1}(r(y))\}$ which is conull by Fubini's theorem.. We define $\varphi_2 : G \to H$ as follows: If $y \in G_2$, then $\varphi_2(y)$ is the constant value to which $\varphi(x)^{-1}\varphi(xy)$ is $\nu_{r(y)}$-almost everywhere equal. If $y \notin G_2$, then $\varphi_2(y) = h$. It is not immediate that φ is Borel, but (a restriction of) it is, as we shall see in a moment.

Set $\Sigma = G_1 \cap G_2$. Then Σ contains the conull set $G_1 \cap B$. We claim that Σ is closed under multiplication. Suppose $(y,z) \in G^{(2)} \cap (\Sigma \times \Sigma)$. Then yz lies in G_1, since G_1 is a contraction. To show that yz lies in G_2, observe first that $\{x | s(x) = r(y)$ and $\varphi(x)\varphi(y)$ is defined and equal to $\varphi(xy)\}$ is a $\nu_{r(y)}$-conull subset of $s^{-1}(r(y))$, since $y \in G_2$. On the other hand,

$$\begin{aligned}\{x|s(x) = r(y) &\text{ and } \varphi(xy)\varphi_2(z) \text{ is defined and equals } \varphi(xyz)\} \\ &= \{x|s(x) = r(z) \text{ and } \varphi(x)\varphi_2(z) \text{ is defined and equals } \varphi(xz)\}y^{-1}\end{aligned}$$

is conull with respect to $\nu_{r(y)}$ on $s^{-1}(r(y))$, since $y \in G_1$ and $z \in G_2$. Hence, as a function of x, $\varphi(x)^{-1}\varphi(xyz)$ is defined and constant $\nu_{r(yz)}$-a.e.on $s^{-1}(r(yz))$ and the value is $\varphi_2(y)\varphi_2(z)$. This shows that yz lies in G_2 *and* that $\varphi_2(yz) = \varphi_2(y)\varphi_2(z)$.

Apply Lemma 4.9 to find a μ-conull set U_0 so that $G_0 := G|_{U_0}$ is contained in Σ and define $\varphi_0 : G \to H$ by the formula

$$\varphi_0(y) = \begin{cases} \varphi_2(y), & y \in G_0 \\ h, & y \notin G_0. \end{cases}$$

Then φ_0 *is* Borel. Indeed, if $y \in G_0$, then $y \in G_2$, and $\varphi_1(y) = \varphi_2(y)$ because $\nu_{r(y)}$ is a probability measure. Since φ_1 is Borel, so is $\varphi_0|G_0$, and since G_0 is Borel, it is clear from the definition of φ_0 that φ_0 is Borel on all of G. Further, $\varphi_0|G_0$ is a homomorphism, since φ_2 is a homomorphism on Σ. The fact that $\varphi_0 = \varphi$ a.e.ν follows from the fact that $\varphi_0 = \varphi_1 = \varphi$ on G_0 and the fact that G_0 is conull. Thus, φ_0 is a weak homomorphism that agrees with φ a.e.C. □

One of the disadvantages of Theorem 4.8 is that there is little control of the contracted groupoid G_0 on which the weak homomorphism, equal a.e.to the given a.e.homomorphism, is an actual homomorphism. It would be nice to assert that given an a.e.homomorphism $\varphi_0 : G \to H$ then there is a homomorphism φ, *defined on all of* G, such that $\varphi_0 = \varphi$ a.e.(This is what we claimed in Theorem 3.20.) This can be done, if, as we are assuming elsewhere in our notes, the underlying groupoid G is σ-compact and, in particular, if the groupoid is a 2nd countable, locally compact groupoid. These discoveries were made by Ramsay in [**161**], and we turn now to a presentation of some of the results found there.

First we need some terminology related to the notion of *similar* homomorphisms and (algebraically) similar groupoids discussed in Chapter 2. See Definitions 2.4 and 2.21.

DEFINITION 4.10. *Suppose that* (G, C) *is a measured groupoid and that, for* $i = 1, 2$, φ_i *is a weak homomorphism from* G *to an analytic Borel groupoid* H. *Let* $U_i \subseteq G^{(0)}$ *be a conull subset such that* $\varphi_i |G|_{U_i}$ *is a homomorphism from* $G|_{U_i}$ *to* H.

(1) *If there is a Borel function* $\theta : G^{(0)} \to H$ *and an inessential contraction* $G|_U$ *of* G, *with* $U \subseteq U_1 \cap U_2$, *such that* $\varphi_2(x) = \theta(r(x))\varphi_1(x)\theta(s(x))^{-1}$ *for all* $x \in G|_U$, *then* φ_1 *and* φ_2 *are called* weakly equivalent *and* θ *is said to implement a* weak equivalence *between* φ_1 *and* φ_2.
(2) *If the set* U *in (1) can be chosen to be invariant, then* φ_1 *and* φ_2 *are called* equivalent *and* θ *is said to implement an* equivalence *between* φ_1 *and* φ_2.

REMARKS 4.11.
1. *If* φ_1 *and* φ_2 *are Borel homomorphisms from* G *to* H *that are similar in the sense of Definition 2.4, and the similarity is implemented by a Borel function* $b : G^{(0)} \to H$, *i.e.,* $\varphi_2(x) = b(r(x))\varphi_1(x)b(s(x))^{-1}$, *for all* $x \in G$, *then* φ_1 *and* φ_2 *are equivalent. Thus, equivalence is a weakening of (Borel) similarity.*
2. *In* [**157**], *'weak equivalence' is called 'similar'.*
3. *The terminology suggests that 'weak equivalence' and 'equivalence' are equivalence relations, and indeed they are, as is easy to see.*
4. *As we noted after Definition 3.24, equivalent representations, defined there, should be called weakly equivalent.*

For the purposes of our discussion here, the advantage of σ-compact spaces in this theory is the presence of Borel cross sections to almost continuous maps. More specifically, we have the following lemma that is a slight reformulation of Lemma 1.1 in [**115**] and is based on a result of Federer and Morse [**65**, Theorem 5.1].

LEMMA 4.12. *Suppose* X *and* Y *are Polish spaces, and* $f : X \to Y$ *is Borel. Suppose also that* A *is a* σ*-compact subset of* X *that may be expressed as the union of a sequence of compact sets* $\{K_n\}_{n=1}^{\infty}$ *on which* f *is continuous. Then there is a Borel function* $g : f(A) \to A$ *such that* $g(f(K_n)) \subseteq K_n$ *for each* n, *and* $f(g(y)) = y$, *for* $y \in f(A)$, *i.e.,* g *is a cross section to* f.

We will apply this lemma in the context of reduction homomorphisms that we define in

DEFINITION 4.13. *Suppose* G *is a Borel groupoid and that* U *is a Borel subset of* $G^{(0)}$. *Suppose also that there is a Borel map* $\theta : G^{(0)} \to G^U$ *such that* $s \circ \theta(u) = u$, $u \in G^{(0)}$, *and such that* $\theta(u) = u$, $u \in U$. *Then the* reduction homomorphism $\psi = \psi_\theta$ *determined by* θ *is defined by the formula*

$$\psi(x) = \theta(r(x)) \cdot x \cdot \theta(s(x))^{-1},$$

$x \in G$.

REMARKS 4.14. *Let* $\psi = \psi_\theta$ *be the reduction homomorphism determined by the Borel map* $\theta : G^{(0)} \to r^{-1}(U)$, *where* $U \subseteq G^{(0)}$ *is a Borel set. Then*

1. *The range of* ψ *is* $G|_U$ *and* $\psi |G|_U$ *is the identity on* $G|_U$. *Further,* ψ *is equivalent to the identity homomorphism in the sense of Definition 4.10. This explains the terminology.*
2. *If* $\varphi : G \to H$ *is a homomorphism, then* $\varphi \circ \theta$ *implements an equivalence between* $\varphi \circ \psi$ *and* φ, *and* φ *and* $\varphi \circ \psi$ *coincide on* $G|_U$.

LEMMA 4.15. [**161**, Lemma 3.1] *Let G be a σ-compact Polish groupoid and let C be an invariant measure class on G, making (G, C) a measured groupoid. Suppose $U_0 \subseteq G^{(0)}$ is a conull Borel set. Then there exist a σ-compact, conull,* invariant *set $U \subseteq G^{(0)}$, and a reduction homomorphism ψ of $G|_U$ onto $G|_{(U \cap U_0)}$.*

PROOF. Since $G^{(0)}$ is the set where r and the identity map agree, and since both are continuous, $G^{(0)}$ is a closed subset of G, and therefore σ-compact. Since $U_0 \subseteq G^{(0)}$ is conull by hypothesis, there is a σ-compact conull subset $U_1 \subseteq U_0$ by [**142**, Theorem 3.2]. Since U_1 is σ-compact and r is continuous, $r^{-1}(U_1)$ is the countable union of closed subsets of G. Consequently, $r^{-1}(U_1)$ is a σ-compact subset of G. But then $U := s(r^{-1}(U_1))$ is σ-compact *and* invariant. Further, the continuous function $s|r^{-1}(U_1)$ has a Borel section $\theta_1 : U \to r^{-1}(U_1)$ by Lemma 4.12. Since U_0 is Borel, the map θ defined by the formula

$$\theta(u) = \begin{cases} u, & u \in U \cap U_0 \\ \theta_1(u), & u \in U \backslash U_0, \end{cases}$$

is also Borel. Moreover, ψ_θ is the desired reduction homomorphism. □

THEOREM 4.16. [**161**, Theorem 3.2] *Let G be a σ-compact Polish groupoid and let C be an invariant measure class on G, making (G, C) a measured groupoid. Let H be an analytic Borel groupoid let $\varphi : G \to H$ be a Borel map.*

1. *If φ is a weak homomorphism, then there is a homomorphism $\varphi_1 : G \to H$ that is equivalent to φ and equals φ a.e. C.*
2. *If φ is an a.e. homomorphism, then there is a homomorphism $\varphi_1 : G \to H$ that equals φ a.e. C.*

PROOF. For the first assertion, recall that by definition (see Definition 4.6), there is a conull set U_0 in $G^{(0)}$ such that $\varphi|G|_{U_0}$ is a homomorphism. Apply Lemma 4.15 to find a σ-compact, conull, invariant subset $U \subseteq G^{(0)}$ and a reduction homomorphism $\psi : G|_U \to G|_{U \cap U_0}$. Choose a unit $u \in H^{(0)}$, arbitrarily, and define $\varphi_1 : G \to H$ by the formula

$$\varphi_1(x) = \begin{cases} \psi(x), & x \in G|_U, \\ u, & x \notin G|_U. \end{cases}$$

Then φ_1 is Borel and it is a homomorphism, since U is invariant. Since $U \cap U_0$ is conull and $\varphi_1 = \psi = \varphi$ on $G|_{U \cap U_0}$, $\varphi_1 = \varphi$ a.e. C. For the second assertion, apply Theorem 4.8 to find a weak homomorphism φ_0 that agrees with φ a.e. C, and then apply the first assertion to find a homomorphism φ that agrees with φ_0 a.e. C. □

We want to caution the reader that Theorem 4.16 is not the panacea one might think it is. Certainly, it solves the problem of having to face a.e. homomorphisms, but what has so far been ignored is the problem of how invariant measure classes are transformed under homomorphisms. This is a very subtle matter, and while we will take it up briefly in Section 5, we will not be able to do justice to the subject in these notes. The interested reader should consult Ramsay's article [**157**] for a thorough treatment.

We conclude this section with a corollary that we promised in our discussion of quasi-invariant measures in Chapter 3.

PROOF OF PROPOSITION 3.15 IN CHAPTER 3.

The proof is taken from Proposition I.3.3 of [**171**]. By Theorem 4.16, it suffices to show that any choice Δ for $d\nu/d\nu^{-1}$ is an a.e. homomorphism. Fix a choice of Δ. We first show that $\Delta(x^{-1}) = \Delta(x)^{-1}$ a.e. ν. Indeed, for all non-negative Borel functions f on G, we have $\int f(x)d\nu(x) = \int f(x)\Delta(x)\,d\nu^{-1}(x) = \int f(x^{-1})\Delta(x^{-1})\,d\nu(x) = \int f(x^{-1})\Delta(x^{-1})\Delta(x)\,d\nu^{-1}(x) = \int f(x)\Delta(x)\Delta(x^{-1})\,d\nu(x)$, which proves that $\Delta(x^{-1})\Delta(x) = 1$ a.e. ν. To show that $\Delta(xy) = \Delta(x)\Delta(y)$ a.e. $\nu^{(2)}$, we show that both $(x,y) \mapsto \Delta(y)$ and $(x,y) \mapsto \Delta(xy)\Delta(x)^{-1}$ are versions of the Radon-Nikodym derivative $d\nu^{(2)}/d(\nu^{(2)})^{-1}$. (Actually, we do not know *a priori* that $\nu^{(2)}$ is absolutely continuous with respect to $(\nu^{(2)})^{-1}$, but this will drop out of the proof. Begin by expanding $\int\int f(x,y)\Delta(y)\,d(\nu^{(2)})^{-1}$ to get

$$\begin{aligned}
\int\int f(x,y)\Delta(y)\,d(\nu^{(2)})^{-1} &= \int\int f(xy,y^{-1})\Delta(y^{-1})\,d\nu^{(2)} \\
&= \int\int\int f(xy,y^{-1})\Delta(y^{-1})\,d\lambda_{r(y)}(x)\,d\lambda^u(y)\,d\mu(u) \\
&= \int\int\int f(x,y^{-1})\Delta(y^{-1})\,d\lambda_{s(y)}(x)\,d\lambda^u(y)\,d\mu(u) \\
&= \int\int f(x,y^{-1})\Delta(y^{-1})\,d\lambda_{s(y)}(x)\,d\nu(y) \\
&= \int\int f(x,y)\Delta(y)\,d\lambda_{r(y)}(x)\,d\nu^{-1}(y) \\
&= \int\int f(x,y)\,d\lambda_{r(y)}(x)\,d\nu(y) \\
&= \int\int\int f(x,y)\,d\lambda_u(x)\,d\lambda^u(y)\,d\mu(u) \\
&= \int\int f(x,y)\,d\nu^{(2)}.
\end{aligned}$$

This shows that $\nu^{(2)}$ is absolutely continuous with respect to $(\nu^{(2)})^{-1}$ and that $(x,y) \mapsto \Delta(y)$ is the Radon-Nikodym derivative $d\nu^{(2)}/d(\nu^{(2)})^{-1}$. To show that $d\nu^{(2)}/d(\nu^{(2)})^{-1}(x,y) = \Delta(xy)\Delta(x)^{-1}$ a.e. , calculate, using the fact that $\Delta(x^{-1}) = \Delta(x)^{-1}$ a.e. and Fubini's theorem, to obtain:

$$\begin{aligned}
&\int\int f(x,y)\Delta(xy)\Delta(x)^{-1}\,d(\nu^{(2)})^{-1} \\
&= \int\int f(xy,y^{-1})\Delta(x)\Delta(xy)^{-1}\,d\nu^{(2)} \\
&= \int\int\int f(xy,y^{-1})\Delta(x)\Delta(xy)^{-1}\,d\lambda_u(x)\,d\lambda^u(y)\,d\mu(u) \\
&= \int\int\int f(xy,y^{-1})\Delta(x)\Delta(xy)^{-1}d\lambda^{s(x)}(y)\,d\lambda_u(x)\,d\mu(u).
\end{aligned}$$

Then change variables $xy \to y$ and write $\nu^{-1} = \int d\lambda_u \, d\mu(u)$ to continue with

$$\begin{aligned}
&\int\int\int f(xy, y^{-1})\Delta(x)\Delta(xy)^{-1} \, d\lambda^{s(x)}(y) \, d\lambda_u(x) \, d\mu(u) \\
&= \int\int f(y, y^{-1}x)\Delta(x)\Delta(y^{-1}) \, d\lambda^{r(x)}(y) \, d\nu^{-1}(x) \\
&= \int\int f(y, y^{-1}x)\Delta(y^{-1}) \, d\lambda^{r(x)}(y) \, d\nu(x) \\
&= \int\int\int f(y, y^{-1}x)\Delta(y^{-1}) \, d\lambda^{u}(y) \, d\lambda^{u}(x) \, d\mu(u) \\
&= \int\int\int f(y, y^{-1}x)\Delta(y^{-1}) \, d\lambda^{r(y)}(x) \, d\lambda^{u}(y) \, d\mu(u) \\
&= \int\int f(y, x)\Delta(y)^{-1} \, d\lambda^{s(y)}(x) \, d\nu(y) \\
&= \int\int f(y, x) \, d\lambda^{s(y)}(x) \, d\nu^{-1}(y) \\
&= \int\int\int f(y, x) \, d\lambda^{s(y)}(x) \, d\lambda_u(y) \, d\mu(u) \\
&= \int\int\int f(y, x) \, d\lambda_u(y) \, d\lambda^{u}(x) \, d\mu(u) \\
&= \int f(y, x) \, d\nu^{(2)}(y, x).
\end{aligned}$$

□

3. Haar Measures for Measured Groupoids

In [**87**], Hahn showed that every analytic measured groupoid has a *measurable* Haar system in the sense spelled out in the following theorem.

THEOREM 4.17. [**87**, Theorem 3.9 ff.] *Let (G, C) be an analytic measured groupoid and let $\nu \in C$ be a symmetric probability measure. Then there is a Borel set $U \subseteq G^{(0)}$ that is conull for $r(C)$ and there is a Borel function $P : G|_U \to (0, \infty)$ such that*

1. *ν has an r-decomposition $\nu = \int \nu^u \, d\mu(u)$ such that $\nu^u(G|_U) = 1$ for all $u \in U$.*
2. *For all non-negative Borel functions f on $G|_U$ and for all $x \in G|_U$,*
$$\int f(xy)P(y) \, d\nu^{s(x)}(y) = \int f(y)P(y) \, d\nu^{r(x)}(y).$$
3. *The map $y \to P(y)/P(y^{-1})$ is a homomorphism from $G|_U$ into $(0, \infty)$.*

Furthermore, if P' and U' have these properties, then there is a Borel function $\phi : U \cap U' \to (0, \infty)$ such that $P'(y) = \phi(s(y))P(y)$ for ν-almost all $y \in G|_{U \cap U'}$.

Thus, if $\lambda^u := P\nu^u|_{G|_U}$, $u \in U$, then $\{\lambda^u\}_{u \in U}$ has the invariance property of a Haar system ($x\lambda^{s(x)} = \lambda^{r(x)}$, $x \in G|_U$), and the support property (supp $\lambda^u = G^u \cap (G|_U)$)*, but the continuity property is replaced with the measurability condition: $u \to \int f \, d\lambda^u$ is Borel for each non-negative Borel function f. Further, if $\underline{\nu} := \int \lambda^u \, d\mu(u)$, $\underline{\nu}$ is equivalent to ν and $y \to P(y)/P(y^{-1})$ is a version of $d\underline{\nu}/d\underline{\nu}^{-1}$.*

DEFINITION 4.18. *The family* $\{\lambda^u\}_{u\in U}$ *is called a* (measurable) Haar system *associated with* ν*, the pair* $(\{\lambda^u\}_{u\in U}, \mu)$ *is called a* Haar measure *for* (G,C) *(associated with* ν*) and* $\Delta := P(y)/P(y^{-1}) = d\underline{\nu}/d\underline{\nu}^{-1}$ *is called its* modular function.

The proof is ingenious and requires numerous delicate calculations with Radon-Nikodym derivatives. In very broad outline, it runs as follows. As we have noted in Remark 4.4, $\nu^{(2)}$ determines a measured groupoid structure on $G^{(2)}$. In particular, $\nu^{(2)}$ is equivalent to $(\nu^{(2)})^{-1}$. In Lemma 3.6 of [**87**], Hahn shows directly that if $\rho = d\nu^{(2)}/d(\nu^{(2)})^{-1}$, then ρ is an a.e.homomorphism of $G^{(2)}$ into the multiplicative group $(0,\infty)$. So, we may apply Theorem 4.8 to find an inessential contraction $G^{(2)}|_U$ on which ρ is a homomorphism. If we knew that $G^{(2)}|_U$ were of the form $G^{(2)}\,|G|_{U_1} = (G|_{U_1})^{(2)}$ for some subset $U_1 \subseteq G^{(0)}$, then we could apply the fact that $(G|_{U_1})^{(2)}$ is cohomologically trivial (See Remark 4.7.) to find a function p such that $\rho(x,y) = p(x)p(xy)^{-1}$ a.e.$\nu^{(2)}$. However, with the aid of von Neumann's Selection Theorem (See [**13**, Proposition I.2.15] or [**10**, Theorem 3.4.3].), Hahn finds a conull subset $U_1 \subseteq G^{(0)}$ such that $(G|_{U_1})^{(2)}$ is conull with respect to $\nu^{(2)}$ and is contained in $G^{(2)}|_U$. This suffices to produce the desired function p. But then the function P of the theorem is almost $p(x^{-1})$. First set $P_1(x) = p(x^{-1})$ and set $\Delta_1(x) = P_1(x)/P_1(x^{-1})$. The fact that function Δ_1 is an a.e.homomorphism requires two and a half pages of calculations, leading ultimately to the conclusion that for all Borel sets E and F in G,

$$\int 1_{E\times F}(x,y)\left(1 - \frac{\Delta_1(xy)}{\Delta_1(x)\Delta_1(y)}\right) d\nu^{(2)}(x,y) = 0.$$

It follows from the Cathéodory Extension Theorem that the map

$$(x,y) \to \left(1 - \frac{\Delta_1(xy)}{\Delta_1(x)\Delta_1(y)}\right)$$

is $\nu^{(2)}$-null and, therefore, that Δ_1 is an a.e.homomorphism. Appealing to Theorem 4.8, U_1 is pared down further so that Δ_1 is equal almost everywhere with respect to $\nu^{(2)}$ to a homomorphism Δ. With this Δ in hand, P_1 has to be adjusted slightly to obtain a new function P so that $\Delta(x) = P(x)/P(x^{-1})$.

The Haar measure $(\{\lambda^u\}_{u\in U}, \mu)$ is associated to a symmetric measure $\nu \in C$. If one changes the measure ν in C one gets a different, but closely related Haar measure as the following proposition indicates. It is a variant of Theorem 3.15.

PROPOSITION 4.19. [**87**, Corollary 3.14] *Suppose for* $i = 1,2$, ν_i *is a symmetric measure in* C *with Haar measure* $(\{\lambda_i^u\}_{u\in U_i}, \mu_i)$ *and suppose* Δ_i *is the associated modular function. Then there is a positive function* ϕ *on* $G^{(0)}$ *such that the following equations hold a.e.with respect to* C :

$$\phi\circ s = (d\nu_2/d\nu_1)(d\mu_1/d\mu_2)\circ r$$

and

$$\frac{\Delta_2}{\Delta_1} = \frac{\phi\circ s}{\phi\circ r}\frac{(d\mu_1/d\mu_2)\circ r}{(d\mu_1/d\mu_2)\circ s}.$$

In particular, Δ_1 *and* Δ_2 *are cohomologous.*

With the existence of a Haar measure on any measured groupoid (G,C) secured, Hahn proceeds to produce a von Neumann algebra $W^*(G,C)$ naturally associated to (G,C) in [**88**]. Throughout the discussion, $(\{\lambda^u\}_{u\in U}, \mu)$ will be a fixed Haar measure for the measured groupoid G and ν will be $\int \lambda^u \, d\mu(u)$. The collection of all Borel functions f on G such that the quantity $\|f\|_I$ is finite will be denoted $L^I(\lambda, \mu)$, where

$$\|f\|_I := \max\left\{ \left\| u \to \int |f(x)| \, d\lambda^u(x) \right\|_\infty, \left\| u \to \int |f(x)| \, d\lambda^u(x) \right\|_\infty \right\},$$

and where the infinity norms are calculated with respect to μ. Thus, $L^I(\lambda, \mu)$ is a measure theoretic analogue of $L^I(G, \lambda)$ defined after Theorem 2.42. A straightforward calculation shows that $L^I(\lambda, \mu)$ is also a Banach $*$-algebra. (See [**88**, Lemma 1.6].) Note that if the groupoid G were cotrivial, i.e., if $G = G^{(0)}$, then $L^I(\lambda, \mu)$ would be the bounded Borel functions on G with the $L^\infty(\mu)$-norm. Thus, it is clear that $L^I(\lambda, \mu)$ is quite a bit bigger than $L^I(G, \lambda)$. A somewhat more laborious calculation, but one that is similar to showing that the left regular representation of a locally compact *group* is contractive in the L^1-norm, shows that the map $L : L^I(\lambda, \mu) \to B(L^2(\nu))$ defined by the formula

$$L(f)\xi(x) = \int f(y)\xi(y^{-1}x) \, d\lambda^{r(x)}(y)$$

is a $*$-representation satisfying the inequality $\|L(f)\| \leq \|f\|_I$. (See [**88**, Corollary 2.2]) Thus, L is the natural extension to $L^I(\lambda, \mu)$ of $\operatorname{Ind} \mu$ defined, in the locally compact setting, on $C_c(G)$. Strictly speaking, of course, $\operatorname{Ind} \mu$ acts on $L^2(\nu^{-1})$ and not on $L^2(\nu)$. However, the map W defined on $L^2(\nu)$ by the formula $W\xi = \xi \cdot \Delta^{\frac{1}{2}}$, where Δ is the modular function of $(\{\lambda^u\}_{u \in U}, \mu)$, is a Hilbert space isomorphism from $L^2(\nu)$ onto $L^2(\nu^{-1})$ that intertwines L and $\operatorname{Ind} \mu$. (See Example 3.28, Definition 3.29, and Exercise 3.30.)

DEFINITION 4.20. *The von Neumann algebra of* (G, C), $W^*(G, C)$, *is defined to be the weakly closed algebra generated by the image of* L.

The definition looks like it depends on the choice of Haar measure $(\{\lambda^u\}_{u \in U}, \mu)$. However, Proposition 4.19 easily implies that the von Neumann algebra constructed with another choice of Haar measure for (G, C) is canonically spatially isomorphic to the von Neumann algebra constructed from $(\{\lambda^u\}_{u \in U}, \mu)$. The following proposition is a summary of some of the analysis in [**88**].

PROPOSITION 4.21. *If* (G, C) *is a measured groupoid with Haar measure* $(\{\lambda^u\}_{u \in U}, \mu)$ *and if* $\nu = \int \lambda^u \, d\mu(u)$, *the von Neumann algebra* $W^*(G, C)$ *acting on* $L^2(\nu)$ *is in standard form. The modular operator* $\underline{\Delta}$ *is given by Schur multiplication by the modular function* Δ *of* $(\{\lambda^u\}_{u \in U}, \mu)$ *and the modular conjugation operator* J *is given by the formula* $J\xi(x) = \overline{\xi(x^{-1})} \, \Delta^{-\frac{1}{2}}(x)$.

If G is locally compact, $\{\lambda^u\}_{u \in G^{(0)}}$ is a Haar system, and if μ is a quasi-invariant measure, then in fact $C_c(G)$ with the inner product coming from $L^2(\nu^{-1})$ is a left Hilbert algebra and $\operatorname{Ind} \mu$ is the left regular representation of this Hilbert algebra [**171**, II.1.10]. In the absence of a topology on G, Hahn builds a modular Hilbert algebra in $L^2(\nu)$ by analyzing the distribution of Δ. One can determine many of the properties of the von Neumann algebra $W^*(G, C)$ in terms of the groupoid and the Haar measure. For example, $W^*(G, C)$ is semi-finite if and only if Δ is a coboundary. More comprehensively, the smooth flow of weights associated with $W^*(G, C)$, in the sense of Connes and Takesaki [**38**] can be described in terms of

Δ. Since this has a very interesting groupoid interpretation, we pause to discuss it here.

This discussion is amplified in [**157**, Section 7]. Let (G, C) be an ergodic groupoid and suppose $\theta : G \to H$ is a homomorphism of G into a locally compact group H. Fix a probability measure $\nu \in C$ with r-decomposition $\nu = \int \nu^u \, d\mu(u)$, and let η be Haar measure on H. Consider the subalgebra $\mathfrak{A}$ of $L^\infty(G^{(0)} \times H, \mu \times \eta)$ consisting of all functions f such that $f(r(x), h) = f(s(x), h \cdot \theta(x))$, for $\nu \times \eta$-almost all $(x, h) \in G \times H$. Then $\mathfrak{A}$ is weak-$*$ closed in $L^\infty(G^{(0)} \times H, \mu \times \eta)$ and invariant under the translation operators $\{T_h\}_{h \in H}$ defined by the formula

$$T_h f(u, k) = f(u, h^{-1}k).$$

One may then apply Theorem 3.3 of [**157**] (which repairs a gap in the main theorem of [**119**]) to conclude that there is an analytic Borel space (S, ω) and an action of H on S, leaving ω quasi-invariant such that if $\tilde{T}_h$ is the action of $h \in H$ on $L^\infty(\omega)$ determined by translation by h on S, then there is an isomorphism from $\mathfrak{A}$ to $L^\infty(\omega)$ carrying T_h to $\tilde{T}_h$. Thus the operators T_h, $h \in H$, are realized by the point transformations on S.

DEFINITION 4.22. *The* range closure *of the homomorphism θ is defined to be the action of H on S with the quasi-invariant measure ω.*

The reason for the terminology is explained in [**123**] and [**157**]. The range closure of a homomorphism is unique up to conjugacy and depends only on the cohomology class of the homomorphism.

THEOREM 4.23. [**88**]*If (G, C) is an ergodic measured groupoid, then the smooth flow of weights on $W^*(G, C)$ is metrically isomorphic to the range closure of Δ regarded as a homomorphism from G to the positive real numbers under multiplication.*

4. Transversals

In [**161**], Ramsay showed that an arbitrary measured groupoid (G, C) is "very close" to being a locally compact groupoid. Specifically, he proved

THEOREM 4.24. [**161**, Theorem 4.1] *Let (G, C) be a measured groupoid. Then there is an inessential contraction $(G|_U, C|_U)$ such that $G|_U$ may be endowed with a second countable, locally compact topology. In particular, $G|_U$ is a σ-compact Polish groupoid.*

The proof of this theorem is modeled on a theorem of Mackey in [**117**] that asserts that if G is an analytic Borel group supporting a σ-finite, right quasi-invariant Borel measure μ, say, then there is a topology on G with respect to which G is a locally compact group and, moreover, μ and Haar measure on G are mutually absolutely continuous. The idea of the proof of Theorem 4.24 is this: Fix a probability measure ν in C and write its s-decomposition as $\nu = \int \nu_u \, d\mu(u)$. As we remarked earlier, one may assume that there is a μ-conull set $X \subseteq G^{(0)}$ with the property that $\nu_{r(x)}x \sim \nu_{s(x)}$ for all $x \in G|_X$ and each ν_u, $u \in X$, is a probability measure. We may also assume that X is Polish and σ-compact by Theorem 3.2 on page 29 of [**142**]. Write G_1 for $G|_X$. Form the Hilbert bundle $X * \mathcal{H}$ defined by the family $\{\nu_u\}_{u \in X}$, as in Example 3.8. Since $\nu_{r(x)}x \sim \nu_{s(x)}$, for all $x \in G_1$, there is a natural injective unitary representation W of G_1 on $X * \mathcal{H}$. Further, it is easy to see that the sets of constant dimension in $X * \mathcal{H}$ are invariant for this representation. It suffices to assume that $X * \mathcal{H}$ is isomorphic to a trivial bundle $X \times H$. One concludes that W is a Borel bijection between G_1 and an analytic subgroupoid, $W(G_1)$, of the Cartesian product of the trivial groupoid on X and the unitary group $\mathcal{U}(H)$ of H, $X \times \mathcal{U}(H) \times X$. Using W to transfer the measure ν to $W(G_1)$, we may apply the result in [**142**], again, to find a conull σ-compact subset B in $W(G_1)$. The groupoid generated by B is conull and σ-compact. By Lemma 4.9 it contains an inessential contraction $W(G_1|_{X_0})$ and using Theorem 3.2 on page 29 of [**142**], once more, X_0 may be assumed to be σ-compact. Then $W(G_1|_{X_0})$ is σ-compact. Now carry the topology on $W(G_1|_{X_0})$ back to $G_1|_{X_0}$ to get a σ-compact Polish topology on $G_1|_{X_0}$. With more work, one can show that X_0 may be chosen so that $G_1|_{X_0}$ becomes locally compact.

One useful consequence of Theorem 4.24 is that it gives a "common domain of repair for a.e.homomorphisms". That is, if (G, C) is a measured groupoid with locally compact inessential contraction $G|_U$, then by Lemma 4.15, every a.e. homomorphism on G is equal a.e. to a homomorphism on $G|_U$. Another is the existence of "countable sections" and "transversals" that we now define.

DEFINITION 4.25. *Let G be a Borel groupoid and let $T \subseteq G^{(0)}$ be a Borel set.*

1. *T is called a* countable section *if and only if T intersects each orbit in at most a countable set.*
2. *For $u \in G^{(0)}$, let ν^u be the measure on G defined by the formula $\nu^u(A) =$ the cardinality of $A \cap G_T^{\{u\}}$, $|A \cap G_T^{\{u\}}|$. The set T is called a* transversal *if there is a sequence of Borel sets $\{A_n\}$ in G that exhaust G, $G = \bigcup A_n$, such that for each n, the function $u \mapsto \nu^u(A_n)$ is bounded.*
3. *T is called* complete *if the smallest invariant subset $[T]$ of $G^{(0)}$ containing T is conull.*

The function $u \to \nu^u$ is an example of what Connes calls a *transverse function* on G [**35**]. In fact, he calls this particular function the *characteristic function* of T. Condition (2) of the definition is what he calls "proper". Observe that if T is a transversal, then for every point $u \in T$, the isotropy group $G|_{\{u\}}$ is countable.

EXAMPLE 4.26. *View the torus $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$ and denote the image of $(\theta, \psi) \in \mathbb{R}^2$ by $[\theta, \psi]$. Fix $\alpha \in \mathbb{R}$ and let $\mathbb{R}$ act on $\mathbb{T}^2$ by the formula $[\theta, \psi] + t = (\theta + t, \psi + \alpha t)$. Then $T := \{[0, \psi] | \psi \in \mathbb{R}\}$ is a transversal for the transformation group groupoid $G = \mathbb{T}^2 \times \mathbb{R}$ and, in fact, T is complete in the strongest sense: $[T] = \mathbb{T}^2$. Note, too, that since $[\theta, \psi] + 1 = [\theta, \psi + \alpha]$, the groupoid $G|_T = \{([0, \psi], n) | n \in \mathbb{Z}\}$, i.e., $G|_T$ is the transformation groupoid associated to the rotation α.*

The following theorem combines Theorems 5.1 and 5.6 in [**161**].

THEOREM 4.27.
1. *If G is a locally compact* principal *groupoid, then there is a transversal T in $G^{(0)}$ such that $[T] = G^{(0)}$.*
2. *If (G, C) is a measured groupoid, then there is a complete countable section in $G^{(0)}$.*

The thing to note about part (1) of this theorem is that no measures are used. In particular, it is not assumed that G has a Haar system. The proof uses ideas of Forrest [**73**] for building lacunary sections for $\mathbb{R}^n$ actions. These, in turn, were inspired by Ambrose's theorem [**2**] that asserts that an ergodic, measure preserving flow is conjugate to a flow built under a function over a measure preserving transformation. Part (2) follows from part (1) by applying Theorem 4.24 to the orbit equivalence relation associated to G.

It should also be noted that without some sort of topological assumptions on the groupoid G, one cannot assert that a transversal T exists such that $[T] = G^{(0)}$. Indeed, using the fact that there is a Borel set X in the plane whose projection p onto the x-axis, say, is *not* Borel (see [**108**]), the Borel equivalence relation $R = \{(x,y) \,|\, p(x) = p(y)\}$, is a standard Borel groupoid which does not have a transversal T satisfying $[T] = R^{(0)}$ " = "X. If such a T were to exist then, as spelled out in [**132**, p.98], the image of X under p would be Borel.

If T is a complete transversal in a measured groupoid (G, C), then one may frequently replace G by $G|_T$, a groupoid with countable orbits and isotropy groups. In general, such groupoids are more easy to handle than groupoids with "continuous" orbits and non-discrete Haar systems. The groupoid $G|_T$ also carries a canonical measure class induced from C, but we will discuss that in the next section. If G is also principal, so that it is effectively an equivalence relation, then $G|_T$ is likewise principal and, after adjusting T a little, as noted in Remark 3.2 of [**66**], may be taken to be standard. Thus, $G|_T$ is effectively a discrete Borel equivalence relation in the sense of

DEFINITION 4.28. *Let X be a standard Borel space and let R be a Borel subset of $X \times X$. Then R is called a* discrete Borel equivalence relation *if R is an equivalence relation with* countable *equivalence classes. If R carries a measure class C making (R, C) a measured groupoid, then we call (R, C) or simply R a* discrete measured equivalence relation.

THEOREM 4.29. [**67**, Theorem 1] *If $R \subseteq X \times X$ is a discrete Borel equivalence relation, then there is a* countable *group G of Borel transformations of X such that $R = \{(x, xt) \,|\, x \in X,\ t \in G\}$ – the orbit equivalence relation determined by G.*

The basic idea of the proof rests on the fact that if Y and X are standard Borel spaces and if $\pi : Y \to X$ is a Borel map and *countable-to-one*, then π maps Borel sets to Borel sets [**108**, Section 39, III, Corollary 5]. Apply [**108**, Section 39, III, Corollary 5] to R and its projections, r and s, onto X to decompose R into a disjoint union of Borel sets, each of which is the graph of a partially defined Borel isomorphism. These may be further refined and pasted together to produce a collection $\mathcal{C}$ of Borel isomorphisms, each mapping X onto X, such that the group generated by $\mathcal{C}$ has R as its orbit equivalence relation.

If R carries a measure class C making it a discrete measured equivalence relation then C is completely determined by a measure μ on X that is quasi-invariant under any choice of discrete group G whose orbit equivalence relation is R. (See Exercise 3.16.) A Haar measure for (R, C), $(\{\lambda^u\}_{u \in U}, \mu)$, is then given by taking λ^u to be counting measure on r-fibers, $\lambda^u(E) = |E \cap r^{-1}(u)|$, $E \subseteq R$, and $u \in X = R^{(0)}$. The von Neumann algebra of a discrete measured equivalence relation was introduced and studied in [**68**], and we will describe a portion of their analysis later. Which chapter?

The collection $\mathcal{C}$ produced by Feldman and Moore has the property that each $\phi \in \mathcal{C}$ has period 2, i.e., ϕ^2 is the identity. Subsequently, Mercer proved and put to good use the fact that for each integer $n > 1$, it is possible to cover R with graphs of transformations of period n [**128**]. Also, the group generated by $\mathcal{C}$ in the Feldman-Moore situation does not act freely (nor do the groups produced by Mercer). This raises the question, first posed in [**67**], of whether every discrete Borel equivalence can be represented as the orbit equivalence relation of a *freely acting* discrete group. In [**1**], Adams gave an example showing that it is not always possible to do this.[3] The analysis of Adams leads one to ask: If R carries an *ergodic* quasi-invariant measure, does there exist a freely acting group whose orbit equivalence relation equals R almost everywhere with respect to the induced measure?

Observe that the collection of transversals in a measured groupoid (G, C) is *σ-ring* that is a σ-algebra only if $r^{-1}(u)$ is countable for each $u \in G^{(0)}$. We write $\mathfrak{T}$ or $\mathfrak{T}(G)$ for the collection of all transversals in (G, C). If G is principal and standard, then $G|_T$ is a discrete Borel equivalence relation for each $T \in \mathfrak{T}$. The following concept was introduced by Connes [**35**] as a generalization of a similarly named concept in the theory of foliations. The discussion following it is, to a large extent, taken from [**132**, p. 102 ff].

DEFINITION 4.30. *Let G be a standard Borel, principal*[4] *groupoid, and let Δ be a homomorphism from G into $(0, \infty)$. A measure μ on $\mathfrak{T}(G)$ is called a* transverse measure with modulus Δ *in case μ is σ-finite and for each $T \in \mathfrak{T}$, $\mu|T$ is quasi-invariant for $G|_T$ with modulus $\Delta|G|_T$. The measure μ is called* invariant *if $\Delta \equiv 1$.*

REMARK 4.31. *A transverse measure with a given modulus is completely determined by its restriction to any transversal T_0 with the property that $[T_0] = G^{(0)}$, assuming, as we shall, that such a transversal exists. Indeed, let μ be a transverse measure with a given modulus Δ and write $\mu|_{T_0}$ for the restriction of μ to T_0. The fact that T_0 is a transversal implies that the source map s restricted to G^{T_0} is countable-to-one. The assumption that $[T_0] = G^{(0)}$ guarantees that s maps G^{T_0} onto $G^{(0)}$. Using* [**108**, Section 39, III, Corollary 5], *one may find a Borel cross section to $s|G^{T_0}$. Following this cross section by r, one obtains a countable-to-one, Borel map f from $G^{(0)}$ to T_0 such that $(f(u), u) \in G$ for all u. Then a calculation shows that μ is given on $\mathfrak{T}(G)$ by the formula*

$$\mu(T) = \int_{T_0} \sum_{\{s \in T | f(s) = t\}} \Delta(t, s)\, d\mu|_{T_0}(t).$$

The meaning of the concept of transverse measure may, perhaps, best be understood through the following observations that are contained in [**132**, pp. 103,104] and in [**35**, pp. 46, 47]. Suppose that X and B are standard Borel spaces and that $p : X \to B$ is a Borel surjection. Let R be the equivalence relation $R = \{(x, y) | p(x) = p(y)\}$. Then R is a standard Borel principal groupoid and B may be identified with the quotient space X/R. We will assume in this discussion that there is a transversal T_0 with $[T_0] = X$. Note that to say that T is a transversal for R means, in particular, that T meets each set $p^{-1}(b)$, $b \in B$, in at most countably many points. Fix a (σ-finite) measure $\tilde{\nu}$ on B and define ν on $\mathfrak{T}$ by the formula

$$\nu(T) = \int_B \left| T \cap p^{-1}(b) \right| \, d\tilde{\nu}(b). \tag{4.1}$$

Then ν is a σ-finite measure on $\mathfrak{T}$. (The reason that ν is σ-finite is that by a Theorem of Kuratowski mentioned above [**108**, Section 39, III, Corollary 5], it is possible to decompose T into a countable family of subsets, $T = \bigcup T_i$, such that

[3] We are grateful to Arlan Ramsay for calling this reference to our attention.

[4] One does not need to assume the groupoid is principal in this discussion, but it makes the exposition easier.

$p|T_i$ is a Borel bijection.) The fact that ν is an invariant transverse measure is now easy to see. Conversely, suppose an invariant transverse measure ν is given on $\mathfrak{T}$, and fix a transversal $T_0 \in \mathfrak{T}$ such that $[T_0] = X$. Thus T_0 is a transversal that maps onto B, via p. Then, using [**108**, Section 39, III, Corollary 5] once more, it is possible to find a Borel set $S_0 \subseteq T_0$ such that on S_0, p is one-to-one and such that $p(S_0) = B$. Let $\tilde{\nu} = p(\nu|S_0)$. Then it is easy to see that with this choice $\tilde{\nu}$, the original measure ν satisfies equation (4.1). The point to be made is that for a general groupoid, if the orbit equivalence relation is nice, i.e. if the quotient space is countably separated, then a transverse measure is essentially the same thing as a measure on the quotient space. However, when the quotient space is bad, which happens when the original groupoid carries a properly ergodic invariant measure class, there are no good measures on the quotient space, but, as it turns out, transverse measures related to the original invariant measure class still exist, as we explain in the next section. See Theorem 4.38.

But first, it may be helpful to digress to explain that Connes defined transverse measures somewhat differently and more generally in [**35**]. His approach enables one to deal smoothly with groupoids that are not necessarily principal. First of all he defined a *transverse function* on a Borel groupoid G to be a family of positive measures $\{\lambda^u\}_{u\in G^{(0)}}$ such that: 1) $u \to \int f(x)\, d\lambda^u(x)$ is measurable for each non-negative measurable function f on G (the values of the integrals may be infinite); 2) each λ^u is supported on G^u; and 3) $x\lambda^{s(x)} = \lambda^{r(x)}$ for all $x \in G$. Thus, a transverse function is a measurable Haar system – but with this important difference: the supports of λ^u need not be all of G^u. The example to keep in mind is the characteristic function of a transversal. (As with characteristic functions of transversals, Connes usually restricts his attention to transverse functions $\{\lambda^u\}_{u\in G^{(0)}}$ that are proper in the sense that G may be expressed as the union of a sequence of sets, $G = \bigcup A_n$, such that $u \to \lambda^u(A_n)$ is bounded for all n.) The collection of all proper transverse functions constitutes a cone $\mathcal{E}^+$ of sections to the bundle of measure spaces $\{M(G^u)\}_{u\in G^{(0)}}$ over $G^{(0)}$. A positive, additive, homogeneous functional Λ on $\mathcal{E}^+$ is called *normal* if $\Lambda(\sup \nu_n) = \sup \Lambda(\nu_n)$ for each increasing sequence $\{\nu_n\}$ in $\mathcal{E}^+$ that is dominated by some $\nu \in \mathcal{E}^+$. Given such a functional and a transverse function $\nu \in \mathcal{E}^+$, one obtains a measure Λ_ν on $G^{(0)}$ defined by the formula $\Lambda_\nu(f) = \Lambda((f \circ s)\nu)$, where f is a non-negative measurable function on $G^{(0)}$. If we write μ_ν for the measure on G defined by the equation $\mu_\nu = \int \nu^u d\Lambda_\nu(u)$ and if $\Delta : G \to (0, \infty)$ is a prescribed homomorphism, then Connes calls Λ a *transverse measure with modulus* Δ in case μ_ν satisfies the equation

$$\Delta \cdot \mu_\nu^{-1} = \mu_\nu,$$

where, recall, μ_ν^{-1} denotes the image of μ_ν under inversion. Thus, a little imprecisely, a transverse measure, according to Connes's definition, gives a consistent way of assigning quasi-invariant measures to supports of transverse functions. (The definition just given is not quite the way Connes defines a transverse measure, but it is equivalent to his, thanks to Théorème 3 of [**35**].) If one restricts a transverse measure Λ to the collection of characteristic functions of transversals as defined in Definition 4.25, one obtains a transverse measure μ^Λ in the sense of Definition 4.30. In fact, as Connes proves in Corollary 6 in [**35**, p. 45], if T is a transversal satisfying $[T] = G^{(0)}$ and if ν is its characteristic function, then $\Lambda \to \Lambda_\nu$ is a bijection between transverse measures with modulus Δ in his sense and measures μ on T that are quasi-invariant under $G|_T$ and have modulus $\Delta|_{G|_T}$.

5. Homomorphisms of measured groupoids. II

In Section 2, we discussed homomorphisms from measured groupoids into arbitrary analytic Borel groupoids. We did not impose any measure structures on the ranges. In order to discuss effectively the problem of relating an invariant measure class on a groupoid to one on a reduction, it is helpful to understand what is at issue concerning homomorphisms between two *measured* groupoids. That is, it is helpful to understand where the problems lie when the range groupoid has an invariant measure class. If (G, C_G) and (H, C_H) are measured groupoids and if $\phi : G \to H$ is a homomorphism, then ϕ carries $G^{(0)}$ to $H^{(0)}$. Moreover, $\phi(G^{(0)})$ may be a null set in $H^{(0)}$. This happens quite commonly, for example, in the case of reductions to transversals. Transversals usually are null sets and so if T is one for G, then the identity map from $G|_T$ to G is a homomorphism whose range on $(G|_T)^{(0)} = T$ is the null set T in $G^{(0)}$. For the general theory, therefore, we want to consider homomorphisms where $\phi(G^{(0)})$ is null, but we do not want $\phi(G^{(0)})$ to be too null. For example, if H were the transformation group groupoid $X \times \Gamma$ determined by a properly ergodic action of a group, Γ, on a measure space X, then we would not want to consider homomorphisms $\phi : G \to X \times \Gamma$ such that $\phi(G^{(0)})$ reduces to a point. These turn out to be too singular. What is necessary is to single out the following important class of subsets of the unit space in a measured groupoid and then to assume that our homomorphisms map these appropriately.

DEFINITION 4.32. *If (G, C) is a measured groupoid, then a subset $E \subseteq G^{(0)}$ is called* negligible *if it is analytic and its saturation $[E]$ is null for $r(C)$.*

Note that the saturation of a set E is $r(s^{-1}(E)) = s(r^{-1}(E))$. Consequently, if E is analytic, so is $[E]$ by [**10**, Corollary 1 to Theorem 3.3.5]. Further, E is negligible iff $[E]$ is negligible, since $[E] = [[E]]$.

Owing to the advances made in [**161**] and presented in Section 2 above, the first part of the following definition is slightly different from the definitions given in [**157**, Definition 4.9 and Definition 6.1]. We adopt the following notation that we shall use throughout the remainder of these notes: If $\phi : G \to H$ is a homomorphism of groupoids, then we write $\tilde{\phi}$ for the restriction of ϕ to $G^{(0)}$. Then $\tilde{\phi}$ is a map from $G^{(0)}$ to $H^{(0)}$.

DEFINITION 4.33. *By a* homomorphism $\phi : G_1 \to G_2$ *between* measured *groupoids, (G_1, C_1) and (G_2, C_2), we shall mean an algebraic homomorphism that is a Borel map, as in Definition 4.6, that also is* regular *or* nonsingular *in the sense that $\tilde{\phi}^{-1}(E)$ is null with respect to $r(C_1)$ for each negligible subset $E \subseteq G_2^{(0)}$. A homomorphism $\phi : G_1 \to G_2$ is called an* isomorphism *if algebraically it is an isomorphism and if it maps the measure class C_1 onto C_2. We will call $\phi : G_1 \to G_2$ a* weak homomorphism *if there is an inessential reduction or contraction $G_1|_{U_1}$ of G_1 such that the restriction of ϕ to $G_1|_{U_1}$ is a homomorphism, i.e., in case ϕ is a weak homomorphism in the sense of Definition 4.6 and the restriction of ϕ to some inessential contraction is regular in the sense just described. A weak homomorphism $\phi : G_1 \to G_2$ is called a* weak isomorphism *in case there are inessential reductions $G_i|_{V_i}$ of G_i, $i = 1, 2$, such that ϕ restricts to an isomorphism of $G_1|_{V_1}$ onto $G_2|_{V_2}$. Also, if $G_1|_{U_1}$ an inessential reduction of G_1 then we shall refer to the restriction of a homomorphism ϕ to $G_1|_{U_1}$ as an* inessential reduction *or* inessential contraction *of ϕ.*

Note that if $\phi : G_1 \to G_2$ is an algebraic homomorphism of groupoids, then for each set $E \subseteq G_2^{(0)}$, $\tilde{\phi}^{-1}([E])$ is an invariant set that contains $\tilde{\phi}^{-1}(E)$. Therefore, $\tilde{\phi}^{-1}([E])$ contains $[\tilde{\phi}^{-1}(E)]$. Thus we may say that a Borel homomorphism ϕ between measured groupoids is regular precisely when the inverse image under $\tilde{\phi}$ of each subset of an invariant null set for $r(C_2)$ is contained in an invariant null set for $r(C_1)$.

It is evident that the composition of homomorphisms is a homomorphism. However, the problem is that the composition of weak homomorphisms may not make sense, and one is often faced with the problem of composing these. Ramsay got around this problem by replacing weak homomorphisms by *weak similarity classes* of weak homomorphisms, in the sense of Definition 4.10 and proving the following fundamental lemma, which is a consequence of von Neumann's selection theorem. Since the proof in [**157**] is a little eliptical, we present one here, thanks to the help of Arlan Ramsay.

LEMMA 4.34. [**157**, Lemma 6.6] *Let $\phi : (G_1, C_1) \to (G_2, C_2)$ be a weak homomorphism between measured groupoids and let V be a complete subset of $G_2^{(0)}$, i.e., assume that $[V]$ is conull with respect to $r(C_2)$. Then there is a weak homomorphism ϕ_0 from G_1 to G_2 that is weakly equivalent to ϕ and whose range is contained in $G_2|_V$.*

PROOF. First, we may as well assume that $G_2^{(0)} = [V]$ because, as is easily seen, $\tilde{\phi}^{-1}([V])$ is a conull invariant set in $G_1^{(0)}$. Indeed, if B is a Borel set in $G_1^{(0)}$ that is disjoint from $\tilde{\phi}^{-1}([V])$, then $\tilde{\phi}(B)$ is an analytic set disjoint from $[V]$ and hence is null with respect to $r(C_2)$. Therefore $\tilde{\phi}(B)$ is negligible and so, then, B is contained in the null set $\tilde{\phi}^{-1}(\tilde{\phi}(B))$. Thus B is null.

Since $G_2^{(0)} = [V]$, s maps G_2^V onto all of $G_2^{(0)}$. By the von Neumann selection theorem [**10**, Theorem 3.4.3], there is an absolutely measurable function $\theta_0 : G_2^{(0)} \to G_2^V$ such that $s \circ \theta_0(u) = u$ for all $u \in G^{(0)}$. In general, one cannot assume that θ_0 is Borel. However, there is a Borel set $V_{00} \subseteq G_2^{(0)}$ that is conull with respect to $\tilde{\phi}(r(C_1)) + r(C_2)$ such that the restriction θ of θ_0 to V_{00} is Borel. Extend θ to all of $G_2^{(0)}$ by setting $\theta(u) = u$ for $u \in G_2^{(0)} \backslash V_{00}$. Then θ is Borel and $\theta^{-1}(G_2^V)$ contains V_{00}. Therefore $V_0 := \theta^{-1}(G_2^V)$ is Borel and conull with respect to $\tilde{\phi}(r(C_1)) + r(C_2)$. Consequently, $U_0 := \tilde{\phi}^{-1}(V_0)$ is a Borel subset of $G_1^{(0)}$ that is null with respect to $r(C_1)$. Further, $G_1|_{U_0}$ is an inessential contraction of G_1, $G_2|_{V_0}$ is an inessential contraction of G_2, and the restriction of ϕ to $G_1|_{U_0}$ is a homomorphism of $G_1|_{U_0}$ into $G_2|_{V_0}$. Thus, we may reduce to U_0 and V_0 and assume that the original selection θ_0, now called θ, is Borel, maps all of $G_2^{(0)}$ to G_2^V and satisfies $s \circ \theta = \iota$.

Let ϕ_0 be the composition of ϕ with the reduction determined by $\theta \circ \phi$, i.e., let $\phi_0(x) = \theta \circ \phi(r(x))\phi(x)\theta \circ \phi(s(x))^{-1}$. (Note that ϕ_0 makes sense, since $s \circ \theta = \iota$.) Then ϕ_0 is a homomorphism of G_1 into G_2 (regularity needs to be checked, but that is easy, since $\tilde{\phi}_0^{-1}(E) = \tilde{\phi}^{-1}(E)$ for every invariant subset of $G_2^{(0)}$) and the range of ϕ_0 is contained in $G_2|_V$, since the range of θ is contained in G_2^V. By construction, ϕ_0 is equivalent to $\theta \circ \phi$ on all of G_1. □

DEFINITION 4.35. [**157**, 6.7, 6.11, 6.12]

1. *Let $\phi : (G_1, C_1) \to (G_2, C_2)$ and $\psi : (G_2, C_2) \to (G_3, C_3)$ be weak homomorphisms. Then ϕ and ψ are* composable *in case there an inessential reduction*

G_1' of G_1 and an inessential reduction G_2' of G_2 such that $\phi(G_1') \subseteq G_2'$, and such that $\psi|G_2'$ is a homomorphism. In this event, we take G_1' so that $\phi|G_1'$ is a homomorphism and then define $\psi \circ \phi$ to be the weak homomorphism such that $\psi \circ \phi|G_1'$ is a homomorphism.

2. *If, for $i = 1, 2$, (G_i, C_i) is a measured groupoid and if $\phi : G_1 \to G_2$ is a weak homomorphism, then we denote by $[\phi]$ the set of all weak homomorphisms ψ from G_1 to G_2 such that ψ is weakly equivalent to ϕ. We write* $\mathrm{Hom}[G_1, G_2]$ *for $\{[\phi] \mid \phi : G_1 \to G_2 \text{ is a weak homomorphism}\}$*
3. *If, for $i = 1, 2$, and 3, (G_i, C_i) is a measured groupoid and if $[\phi] \in \mathrm{Hom}[G_1, G_2]$ and $[\psi] \in \mathrm{Hom}[G_2, G_3]$, then $[\psi] \circ [\phi]$ is defined to be $[\psi_1 \circ \phi_1]$, where $[\psi_1] = [\psi]$, $[\phi_1] = [\phi]$, and ψ_1 and ϕ_1 are composable.*

A systematic application of Lemma 4.34 shows that these notions are well defined and that the composition of weak equivalence classes of weak homomorphisms is associative (see Section 6 of [**157**]). It might be helpful to note that these notions are reminiscent of the concept of germs of continuous maps and their algebraic properties; the notation even suggests this analogy. The difference is that rather than looking at equivalence classes of maps in neighborhoods of points, focusing on smaller and smaller sets, one declares maps to be the equivalent if they essentially agree everywhere, i.e., behavior at infinity is what is important.

DEFINITION 4.36. *Given measured groupoids (G, C_G) and (H, C_H), we shall call a homomorphism $\phi : G \to H$ a* strict similarity*, and say that G and H are* strictly similar*, in case there is a homomorphism $\psi : H \to G$ such that $\psi \circ \phi$ is equivalent to ι_G on all of G (see part (2) of Definition 4.10) and such that $\phi \circ \psi$ is equivalent to ι_H on all of H. Likewise, we call a weak homomorphism $\phi : G \to H$ a* similarity*, and say that G and H are* similar*, in case there is a weak homomorphism $\psi : H \to G$ such that $[\phi] \circ [\psi] = [\iota_H]$ and $[\psi] \circ [\phi] = [\iota_G]$.*

The notions of similarity and strict similarity for groupoids are, of course, equivalence relations. Two locally compact groups are similar iff they are strictly similar, and this happens if and only if they are isomorphic. As we noted above, Mackey called a similarity class of ergodic measured groupoids a virtual group (see [**120, 123**]). He shows in [**123**] that when two groupoids are given by transitive actions of a locally compact group, then the two transformation group groupoids are similar if and only if the subgroups coming from a choice of stabilizers in the two phase spaces are conjugate. Thus he viewed a virtual group is a generalization of a conjugacy class of closed subgroups of a locally compact group.

REMARK 4.37. *It is worthwhile to point out that if (G_i, C_i), $i = 1, 2$, are measured groupoids, if (H, C_H) is a third measured groupoid, and if $\phi : G_1 \to G_2$ is a similarity, then one obtains a bijection $\phi^* : \mathrm{Hom}[G_2, H] \to \mathrm{Hom}[G_1, H]$ via composition: $\phi^*([\psi]) = [\psi] \circ [\phi]$. Further, if H is a locally compact abelian group, then under the evident pointwise operations, the spaces* $\mathrm{Hom}[G_i, H]$ *become abelian groups and ϕ^* is a group isomorphism. The set* $\mathrm{Hom}[G, H]$ *is also called the* first cohomology space of G with coeficients in H *and is then denoted $H^1(G, H)$. If H is abelian, higher cohomology groups, $H^n(G, H)$, $n \geq 1$, may also be defined and we will say a little about this later. However, groupoid cohomology is an involved subject that we shall leave largely untouched in these notes.*

References to Moore, Westman, and Ramsay. Others? Kumjian and Renault?

Notice in particular that if (G, C) is a measured groupoid and if $H = (0, \infty)$ under multiplication, then we get a well-defined element $[\Delta]$ of $H^1(G, H)$ associated *with the modular function* Δ *of any Haar measure for* G. *Owing to Proposition 4.19, the class* $[\Delta]$ *is independent of the choice of Haar measure for* G*; i.e., it depends only on the class* C. *Consequently, we shall call* $[\Delta]$ the modular function *of* C.

The following result, due to Ramsay, is a key result in the theory. It shows how to give a good measure class to the reduction of a measured groupoid to a transversal.

THEOREM 4.38. [**157**, Theorem 6.17] *If* (G, C) *is a measured groupoid and if* V *is a Borel subset of* $G^{(0)}$ *such that* $[V]$ *is conull, then there is a canonical measure class* $C|_V$ *on* $G|_V$ *such that* $(G|_V,\ C|_V)$ *is a measured groupoid that is similar to* (G, C).

OUTLINE OF PROOF. First, we may as well assume that $G^{(0)} = [V]$. Then s maps G^V onto all of $G^{(0)}$. By the von Neumann selection theorem [**10**, Theorem 3.4.3], there is an absolutely measurable function $f : G^{(0)} \to G^V$ such that $s \circ f(u) = u$ for all $u \in G^{(0)}$. As before, one cannot assume that f is Borel. However, it differs from a Borel function defined on a Borel set U, say, at most on a set of measure zero. So, we may assume $f : U \to G^V$ is a Borel function such that $s \circ f(u) = u$ a.e.$r(C)$ on the conull Borel set U. Further, since V is Borel, we may assume that U contains V and that f is the identity on V. Then the set $U_0 := \{u \in G^{(0)} | \, s \circ f(u) = u\}$ is Borel, conull and contains V. Thus, we may reduce to U_0 and assume that there is a Borel function $f : G^{(0)} \to G^V$ such that $f \circ s = \iota$ and such that $f = \iota$ on V. The point to keep in mind is that if even if at the outset we had assumed that $G^{(0)} = [V]$, we still would have had to reduce to U_0 which could very well be proper.

Now let $\psi : G \to G|_V$ be the reduction homomorphism determined by f: $\psi(x) = f(r(x))xf(s(x))^{-1}$, $x \in G$. Then ψ is algebraically a homomorphism from G onto $G|_V$ that certainly is a Borel map. The measure class $C|_V$ on $G|_V$ is taken to be the image of C under ψ. A page and a half of arguments show that $C|_V$ is an invariant measure class (see [**157**, pp. 291,292]) so that $(G|_V, C|_V)$ is a measured groupoid.

By definition of $C|_V$, ψ is a homomorphism of the measured groupoid G to the measured groupoid $G|_V$. Moreover, by the definitions of f and ψ, if ϕ denotes the identity map from $G|_V$ into G, then $\psi \circ \phi = \iota_{G|_V}$ and $\phi \circ \psi = \psi$, while f implements an equivalence between ψ and i_G on all of G. This shows that (G, C) is strictly similar to $(G|_V, C|_V)$ (assuming that the cross section f is defined on all of $G^{(0)}$). □

This result secures the relation we were seeking in the preceding section between an invariant measure class C on a groupoid G and a transverse measure on its σ-ring of transversals. What appears to get lost in the process is the precise relation between the modular function for a Haar measure giving C and the modular function for a Haar measure in $C|_T$, for a complete transversal T. However, similar groupoids have isomorphic cohomology as we indicated in Remark 4.37 and since the modular homomorphism of a measure class does not depend on any measure that represents it, it follows that up to weak equivalence a weak similarity between G and $G|_T$ carries the modular function for any Haar measure giving C to the modular funciton for any prescribed Haar measure for $C|_T$.

REMARK 4.39. *It is worth emphasizing again that given a measured groupoid* (G, C) *and a Borel set* $V \subseteq G^{(0)}$ *such that* $[V]$ *is conull, even one with* $G^{(0)} = [V]$, *the measured groupoids* (G, C) *and* $(G|_V, C|_V)$*, while similar, may not be strictly similar. One situation, however, where strict similarity does occur is this: Assume that* G *and* V *are* σ*-compact (and that* $G^{(0)} = [V]$*). Then* G^V *is* σ*-compact and we may apply Lemma 4.12 to produce a cross section* $f : G^{(0)} \to G^V$ *to* s *satisfying the conditions in the proof of Theorem 4.13. By that proof, then,* f *yields the desired strict similarity between* (G, C) *and* $(G|_V, C|_V)$*. In particular, this applies when* G *is locally compact and* V *is closed, or even* F_σ*.*

Make connection with Putnam and Spielberg's work on Smale groupoids.

It is also worthwhile to point out that similarities between measured groupoids have a very concrete representation, as the following result shows.

THEOREM 4.40. [**66, 161**] *Let* ϕ *be a similarity from the measured groupoid* (G, C_G) *to the measured groupoid* (H, C_H)*. Then there are complete subsets* $U \subseteq G^{(0)}$ *and* $V \subseteq H^{(0)}$ *such that* ϕ *may be written as the composition,* $\phi = \iota \circ \theta \circ \rho$*, where* ρ *is a reduction of (an inessential contraction of)* G *to* $G|_U$*,* $\theta : G|_U \to H|_V$ *is an isomorphism, and where* ι *is the imbedding of* $H|_V$ *into* H*.*

The proof is based on the fact that G and H have inessential reductions that carry locally compact topologies [**161**, Theorem 5.6]. The topologies are used to guarantee the existence of complete countable sections, see Theorem 4.27. These are used as in [**66**, Theorem 5.5] to complete the proof. The theorem is reminiscent of the well-known fact that an isomorphism between von Neumann algebras may be expressed as the composition of a reduction, a spatial isomorphism, and an ampliation. This analogy is further strengthened by the next two theorems which show, among other things, that measured groupoids with "continuous orbits" are groupoid theoretic analogues of infinite von Neumann algebras. If (G, C) is a measured groupoid, then it is said to have *continuous orbits* if for some (and therefore any) probability measure $\nu \in C$, $r(\nu)$-almost all the measures in the r-decomposition of ν are continuous. In such groupoids (almost) all orbits are uncountable. In the following theorem, we write I^2 for the trivial measured groupoid, $[0, 1] \times [0, 1]$ with area measure.

THEOREM 4.41. [**66**, Corollary 5.8 and Theorem 4.8] *Let* (G, C_G) *and* (H, C_H) *be measured groupoids and let* $\phi : G \to H$ *be a similarity.*

1. *If* G *and* H *have continuous orbits, then the class of* ϕ *contains a weak isomorphism.*
2. *If* G *is isomorphic to* $G \times I^2$ *and if* H *is isomorphic to* $H \times I^2$*, then the equivalence class of* ϕ *contains a weak isomorphism.*

As indicated, the theorem is the combination of two results of Feldman, Hahn, and Moore in [**66**, Corollary 5.8 and Theorem 4.8], but in a slightly strengthened form, thanks to the results in [**161**]. See Ramsay's Theorem 6.1 in particular. Part (2) is a kind of stabilization theorem that we shall meet again in the context of Morita equivalence of groupoids.

Link back to this in the next chapter.

THEOREM 4.42. *Let* (G, C) *be a principal measured groupoid with continuous orbits, then for any complete transversal* T *in* $G^{(0)}$*,* G *is weakly isomorphic to* $G|_T \times I^2$*. Further,* $W^*(G, C)$ *is spatially isomorphic to* $W^*(G|_T, C|_T) \otimes B(L^2(I))$*.*

The first assertion is essentially Theorem 5.6 of [**66**] coupled with the fact that every principal measured groupoid is orbitally concrete, Theorem 4.40. The second assertion follows from the first since a weak isomorphism between groupoids implements a spatial isomorphism between their von Neumann algebras.

CHAPTER 5

Morita Theory and Equivalence of Groupoids

Morita theory has played a major role in operator algebra for over 20 years. It is natural to wonder how this notion is reflected in terms of coordinates. Explaining how is the principal goal of this chapter. We begin by reviewing some of the salient features of Morita theory for C^*-algebras in Section 1. These will provide models for the groupoid notions. We present more material than we shall need for this chapter, but the excess will be used later. Then we take up the notion of equivalent groupoids in Section 2. In Section 3, we prove that equivalent groupoids have strongly Morita equivalent C^*-algebras. The details of the proof are complete except for one proposition, whose proof will be given in Section 4. This proposition has as special case Lemma 3.31 in Chapter 3 that played a crucial role in the proof of Theorem 3.32. Section 5 is devoted to analyzing when the imprimitivity groupoid associated with a principal G-space has a Haar system. Additional examples and extensions of the theory will be presented in Chapter 7.

1. Morita Theory for C^*-Algebras

Morita theory for rings was invented by K. Morita in 1958 [**133**], although the basic notions of the theory were known much earlier. The idea is that two rings should be identified, i.e., declared equivalent, if they have isomorphic categories of modules. Thus, for example, for each pair of integers m and n the algebras $M_m(\mathbb{C})$ and $M_n(\mathbb{C})$ should be considered equivalent and they all should be considered equivalent to the field of scalars $\mathbb{C}$ itself. In 1974, Rieffel realized the relevance of this idea for the theory of induced representations in [**177**]. The key construct involved is what has become known as a Hilbert C^*-module. The theory of these, in turn, may be traced back to Kaplansky's paper [**96**]. Kaplansky considered modules only over commutative C^*-algebras, and very little was done with them until Paschke's paper [**143**] and Rieffel's investigation appeared. Now they are ubiquitous in the operator algebra literature. We will present the definition and some of the basic facts about them that we shall need, but we shall refer the reader to two recent texts where details are presented in a very accessible manner, [**109**] and [**156**].

DEFINITION 5.1. *Let A be a C^*-algebra and let X be a right module over A. We call X a* right Hilbert C^*-module *over A in case there is a sesquilinear map $\langle\cdot,\cdot\rangle : X \times X \to A$, that is conjugate linear in the* first *variable, such that*

1. $\langle x, ya\rangle = \langle x, y\rangle a$, $x, y \in \mathsf{X}$, $a \in A$.
2. $\langle x, y\rangle^* = \langle y, x\rangle$, $x, y \in \mathsf{X}$.
3. $\langle x, x\rangle \geq 0$ *in A and* $\langle x, x\rangle = 0$ *only when* $x = 0$.

A left Hilbert C^*-module *is defined in the same way, except that X is required to be a left A-module, the pairing $\langle\cdot,\cdot\rangle$ is required to be conjugate linear in the second variable, and the equation $\langle ax, y\rangle = a\langle x, y\rangle$ replaces the first condition above.*

REMARK 5.2. *Occasionally, we will begin with a module* X_0 *over a pre-*C^**-algebra* A_0 *that is endowed with an* A_0*-valued inner product* $\langle\cdot,\cdot\rangle$ *satisfying the three conditions of Definition 5.1. The third condition in this context means that* $\langle x,x\rangle$ *may be expressed as* a^*a *for some a in the* completion A *of* A_0*. Proposition 2.3 of* [**143**] *(and Proposition 2.9 of* [**177**]*) imply that the expression* $\|x\| := \|\langle x,x\rangle\|_{A_0}^{1/2}$ *is a norm on* X_0*. The completion,* X*, of* X_0 *with respect to this norm becomes a Hilbert* C^**-module over the completion* A *of* A_0 *in an obvious way. Unless explicitly stated to the contrary, we shall assume that our modules are complete in the norm just given and when we must consider incomplete modules, we shall use the prefix "pre-", i.e., we shall speak of pre-Hilbert* C^**-modules over pre-*C^**-algebras.*

REMARK 5.3. *Paschke* [**143**, Definition 2.1] *calls a Hilbert* C^**-module over* A *a Hilbert* A*-module. He works on the right, but his inner product is conjugate linear in the second variable and he assumes the equation* $\langle xa,y\rangle = \langle x,y\rangle a$ *to compensate. Rieffel* [**177**, Definition 2.8] *calls a Hilbert* C^**-module over* A *a right or left* A-rigged space. *When these lectures were given, we called Hilbert* C^**-modules* Hermitian operator modules. *We felt that 'Hilbert* C^**-module' was too close to another concept that is in common use, Hilbert module (see Chapters 9 and 10), and we felt that the emphasis on the term 'operator module' was more in tune with the theory of operator spaces that has become a major industry in recent years. However, 'Hilbert* C^**-module' seems to have become the accepted term and we shall use it in these notes.*

A word about notation may prove helpful: We shall consistently use sans serif letters to denote Hilbert C^**-modules.*

Examples of Hilbert C^*-modules are easy to come by. The following list essentially exhausts all the ones we shall use in these notes, thanks to Kasparov's Stabilization Theorem (Theorem 5.9 below). Despite the apparent simplicity of the examples, the theory is quite rich, as we shall see.

EXAMPLES 5.4.
1. *Let* $\mathsf{X} = C_n(A)$ *denote the collection of all n-tuples,* $n < \infty$*, from* A*. These form a right Hilbert* C^**-module over* A *with the obvious right action and inner product* $\langle (x_n),(y_n)\rangle = \sum x_n^* y_n$*. The space* $\mathsf{X} = C_\infty(A)$ *is defined in essentially the same way: it is the collection of all sequences* (x_n) *such that the series* $\sum x_n^* x_n$ *converges in* A*. For two such sequences,* (x_n) *and* (y_n)*, the sum* $\sum x_n^* y_n$ *converges in* A *and defines the inner product* $\langle (x_n),(y_n)\rangle$*. The spaces* $C_n(A)$ *and* $C_\infty(A)$ *are called* column space over A *(of dimension* n*,* $n \le \infty$*). The reason for this is that elements in* $C_n(A)$ *may be viewed as* $n \times 1$ *matrices over* A *and the inner product of* $x,y \in C_n(A)$*,* $\langle x,y\rangle$*, is* $\langle x,y\rangle = x^*y$*, where the latter product is matrix multiplication. In the literature, what we are calling column space over* A *is usually called Hilbert space over* A*. We find the column space terminology more compelling, however. It calls attention to the operator space structure* I.O.U. *on Hilbert* C^**-module. We will have more to say about this later.*
2. *More generally, the direct sum of any number of Hilbert* C^**-modules is a Hilbert* C^**-module in an obvious way. Kasparov's Stabilization Theorem (Theorem 5.9) asserts that under very mild separability hypotheses every Hilbert* C^**-module is a direct summand of* $C_n(A)$ *for some* $n \le \infty$*.*
3. *If* X *is a right ideal in* A*, then* X *becomes a Hilbert* C^**-module with inner product* $\langle x,y\rangle = x^*y$*.*

4. *Let A be a C^*-algebra and suppose B is a subalgebra. (In the non-unital case, we can extend this discussion in a natural way to allow B to be a subalgebra of the multiplier algebra of A.) Let $P : A \to B$ be a linear map that satisfies following conditions:*
   (a) $P(a^*) = P(a)^*$, $a \in A$.
   (b) *P is positive in the sense that $P(a^*a) \geq 0$ in B for each $a \in A$.*
   (c) $P(ab) = P(a)b$, *for all $a \in A$ and for all $b \in B$.*

   Then if we define $\langle a_1, a_2 \rangle = P(a_1^* a_2)$, it is easy to see that the conditions of Definition 5.1 are satisfied, except that $\langle a, a \rangle$ might be zero for some non-zero a. Dividing out by the elements of "length" zero, we obtain a (usually incomplete) Hilbert C^*-module over B.

REMARK 5.5. *Maps of the form P are called (*conditional*)* expectations*. Usually, they are only introduced in the unital setting and then it is assumed that the identity of A is the identity of B and that P is unital. This assumption guarantees that the inner product it determines is* full *in the sense that the span of the elements $\langle a_1, a_2 \rangle$ is dense in B. In general, if X is a Hilbert C^*-module over a C^*-algebra A, then the span of the elements of the form $\langle x, y \rangle$, $x, y \in \mathsf{X}$, is an ideal in A called the* support of X. *We say that the module is* full *if its support is dense in A.*

As a normed linear space, a Hilbert C^*-module X carries an algebra of bounded (i.e. continuous) linear transformations, $B(\mathsf{X})$. However, we will want to restrict our attention to those that are module maps, i.e., $T(xa) = T(x)a$, *and* we will want to restrict our attention further to those module maps $T \in B(\mathsf{X})$ for which there is another operator $T^* \in B(\mathsf{X})$ that satisfies the equation $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathsf{X}$. Such a T is called *adjointable* and we write $\mathbb{B}(\mathsf{X})$ for the collection of all adjointable operators on X.

Actually, as Lance points out on page 8 of [**109**], if a map T (that *a priori* is not even linear) has an adjoint T^*, i.e., if T^* satisfies $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathsf{X}$, then T and T^* are module maps and continuous. The fact that not every element of $B(\mathsf{X})$ need be in $\mathbb{B}(\mathsf{X})$ is seen from the following example due to Paschke.

EXAMPLE 5.6. [**143**, p. 447] *Let $A = C([0,1])$ and let J be the ideal $C_0((0,1])$. If $\mathsf{X} = J \oplus A$, with $\langle a \oplus b, a_1 \oplus b_1 \rangle = a^*a_1 + b^*b_1$, and if T is defined by the formula $T(a,b) = (0,a)$, then $T \in B(\mathsf{X})$, but not in $\mathbb{B}(\mathsf{X})$.*

A special subcollection of $\mathbb{B}(\mathsf{X})$ needs to be singled out. For $x, y \in \mathsf{X}$, we write $x \otimes y^*$ for the "rank one" linear transformation on X defined by the formula $x \otimes y^*(z) = x\langle y, z \rangle$. Then it is not difficult to see that $x \otimes y^*$ is adjointable, with $(x \otimes y^*)^* = y \otimes x^*$. We write $\mathbb{K}(\mathsf{X})$ for the closed linear span in $\mathbb{B}(\mathsf{X})$ of all the $x \otimes y^*$ as x and y run through X. Elements of $\mathbb{K}(\mathsf{X})$ are called *compact* operators on X, although they are usually not compact in the traditional sense of the word.

The formulas we have given are presented under the assumption that X is a right Hilbert C^*-module and then we write elements of $\mathbb{B}(\mathsf{X})$ and $\mathbb{K}(\mathsf{X})$ on the left. If X is a *left* Hilbert module over A, we will write operators on the right and for x and y in X we will write $x^* \otimes y$ for the operator given by the equation $(z)x^* \otimes y = \langle z, x \rangle y$. The following proposition covers most of the basic things we need to know about $\mathbb{B}(\mathsf{X})$ and $\mathbb{K}(\mathsf{X})$. The proof may be assembled easily from propositions in [**109**].

PROPOSITION 5.7. *If X is a (right) Hilbert C^*-module over a C^*-algebra A, then $\mathbb{B}(\mathsf{X})$ and $\mathbb{K}(\mathsf{X})$ are C^*-algebras, with $\mathbb{B}(\mathsf{X})$ equal to the multiplier algebra of*

$\mathbb{K}(\mathsf{X})$. *Further,* X *becomes a* left *Hilbert* C^**-module over* $\mathbb{K}(\mathsf{X})$*, where the* $\mathbb{K}(\mathsf{X})$*-inner product is defined by the formula*

$$_{\mathbb{K}(\mathsf{X})}\langle x, y\rangle = x \otimes y^*,$$

and $\mathbb{K}({}_{\mathbb{K}(\mathsf{X})}\mathsf{X})$ *is naturally isomorphic to* A *through the formula*

$$x^* {}_{\mathbb{K}(\mathsf{X})}\otimes y \to \langle x, y\rangle_A.$$

There is an obvious version of this proposition that begins with a *left* Hilbert C^*-module over the C^*-algebra A. For reasons that we shall explain later (See Theorem 5.23.), $\mathbb{K}(\mathsf{X})$ is also called the *imprimitivity algebra* of the Hilbert C^*-module X.

EXAMPLE 5.8. *If* $\mathsf{X} = C_n(A)$*, with* $n < \infty$*, and if* A *is unital, then* $\mathbb{B}(\mathsf{X}) = \mathbb{K}(\mathsf{X}) = M_n(A) = M_n \otimes A$*, where we have used equality signs for what really are natural identifications. If* $\mathsf{X} = C_\infty(A)$*, then* $\mathbb{K}(\mathsf{X}) = K \otimes A$*, where* K *denotes the compact operators on an infinite dimensional separable Hilbert space. There is no issue about what tensor product we are using here, since* $M_n(\mathbb{C})$ *and* K *are nuclear. If* A *is unital and* $\mathsf{X} = C_\infty(A)$*, it is still useful to think of* $\mathbb{B}(\mathsf{X})$ *as* $B(H) \otimes A$*, but care must be taken since* $B(H)$ *is not nuclear.*

We have discussed operators on a fixed Hilbert C^*-module, but the extension of the notions of adjointable operators and compact operators to operators between two, possibly different, Hilbert C^*-modules is straightforward and clear. Also, an *isomorphism* $U : \mathsf{X} \to \mathsf{Y}$ between two Hilbert C^*-modules is nothing but a module map from X onto Y such that $\langle Ux, Uy\rangle = \langle x, y\rangle$, for all $x, y \in \mathsf{X}$. Such a map is easily seen to be isometric. The following result was promised above and shows that our list of examples covers most bases.

THEOREM 5.9. *If* X *is a Hilbert* C^**-module over a* C^**-algebra* A *that is countably generated as a module over* A *(i.e. if there is a countable subset of* X *such that the closed* A*-module it generates is all of* X*), then there is a projection* $P \in \mathbb{B}(C_\infty(A))$ *such that* X *is isomorphic to* $PC_\infty(A)$*.*

We like to think of modules of the form $C_n(A)$, $n \leq \infty$, as the operator analogue of *free* modules. Kasparov's theorem says, then, that all (countably generated) Hilbert C^*-modules over a C^*-algebra are projective. This reinforces the idea that we shall develop in Chapters 9 and 10 that C^*-algebras should be viewed as the infinite dimensional analogue of finite dimensional semisimple algebras.[1]

DEFINITION 5.10. *Let* A *and* B *be* C^**-algebras, and let* X *be an* A*-*B *bimodule. Then* X *is called an* A-B-equivalence bimodule *(or an* A-B-imprimitivity bimodule*) in case the following conditions are satisfied:*

1. X *is endowed with* A*- and* B*- valued inner products making* X *a left Hilbert* C^**-module over* A *and a right Hilbert* C^**-module over* B*.*
2. *The inner products are linked by the formula*

$$_A\langle x, y\rangle z = x\langle y, z\rangle_B,$$

for all $x, y, z \in \mathsf{X}$*.*

[1] In algebra, one proves that a finite dimensional algebra is semisimple if and only if each of its modules is projective.

3. *The action of A on X_B is contractive in the B-norm and the action of B on ${}_A\mathsf{X}$ is contractive in the A-norm, i.e., $\langle ax, ax\rangle_B \le \|a\|^2\langle x,x\rangle_B$ and ${}_A\langle xb, xb\rangle \le \|b\|^2{}_A\langle x,x\rangle$.*
4. *X has full support in both A and B.*

Further, we shall say that A and B are strongly Morita equivalent *(abbreviated* SME*) in case there is an A-B-equivalence bimodule.*

Of course, we need to prove that strong Morita equivalence is an equivalence relation. We will indicate why this is the case, in a moment, but first observe that if X is a Hilbert C^*-module over a C^*-algebra A (either right or left), then A and $\mathbb{K}(\mathsf{X})$ are strongly Morita equivalent. The following easily proved proposition is a converse assertion that helps to anchor the notion of strong Morita equivalence.

PROPOSITION 5.11. *Suppose X is an A-B-equivalence bimodule. Then the map from $\mathbb{K}(\mathsf{X}_B)$ to A defined by the formula*

$$x \otimes y^* \to {}_A\langle x, y\rangle$$

is a C^*-isomorphism. Likewise the map from $\mathbb{K}({}_A\mathsf{X})$ to B defined by the formula

$$x^* \otimes y \to \langle x, y\rangle_B$$

is a C^*-isomorphism.

Thus, two C^*-algebras are strongly Morita equivalent if and only if one can be realized as the compact operators of a Hilbert C^*-module over the other.

The following example ties the opening remarks of this section to the definition of strong Morita equivalence.

EXAMPLE 5.12. *For positive integers m and n, let $M_{m,n}(\mathbb{C})$ denote the collection of all $m \times n$ complex matrices. Then $M_{m,n}(\mathbb{C})$ is evidently an $M_m(\mathbb{C})$-$M_n(\mathbb{C})$-bimodule in the algebraic sense. But it is easy to see that $M_{m,n}(\mathbb{C})$ is an $M_m(\mathbb{C})$-$M_n(\mathbb{C})$-equivalence bimodule with inner products given by the formulae:*

$${}_{M_m(\mathbb{C})}\langle S, T\rangle = ST^*$$

and

$$\langle S, T\rangle_{M_n(\mathbb{C})} = S^*T,$$

$S, T \in M_{m,n}(\mathbb{C})$.

REMARK 5.13. *If X is an A-B-equivalence bimodule, then X carries two norms, one from the A-valued inner product, the other from the B-valued inner product. Thanks to the second condition in Definition 5.10, or to Proposition 5.11, the two norms are identical.*

The following proposition is easy to verify and the proof will be omitted.

PROPOSITION AND DEFINITION 5.14. *Let X be a* left *Hilbert C^*-module over a C^*-algebra A. Let $\tilde{\mathsf{X}}$ denote the additive group X with the conjugate action of $\mathbb{C}$ and the "right starred" action of A: $c\tilde{x} := \widetilde{(\bar{c}x)}$, $c \in \mathbb{C}$, and $\tilde{x}a := \widetilde{(a^*x)}$, $a \in A$, where we write $\tilde{x}$ for x viewed as an element of $\tilde{\mathsf{X}}$. Also, give $\tilde{\mathsf{X}}$ the A-valued inner product: $\langle \tilde{x}, \tilde{y}\rangle_A := {}_A\langle y, x\rangle$. Then $\tilde{\mathsf{X}}$ is a* right *Hilbert C^*-module over A called the* dual X. *Further, $\tilde{\mathsf{X}}$ is isomorphic to $\mathbb{K}({}_A\mathsf{X}, A)$ when we identify $\tilde{x}$ with the map defined by the formula: $\tilde{x}(y) = {}_A\langle x, y\rangle$.*

Similarly, if X *is a* right *Hilbert* C^**-module over* A*, then* $\tilde{\mathsf{X}}$ *becomes a* left *Hilbert* C^**-module over* A*, isomorphic to* $\mathbb{K}(\mathsf{X}_A, A)$*, under the operations:* $c\tilde{x} := \widetilde{(\bar{c}x)}$, $c \in \mathbb{C}$, $a\tilde{x} := \widetilde{(xa^*)}$, $a \in A$*, and* ${}_A\langle \tilde{x}, \tilde{y}\rangle := \langle y, x\rangle_A$.

PROPOSITION AND DEFINITION 5.15. *Suppose that* X *is a right Hilbert module over the* C^**-algebra* A*, that* Y *is a right Hilbert* C^**-module over the* C^**-algebra* B*, and that there is a* $*$*-homomorphism* ϕ *of* A *into* $\mathbb{B}(\mathsf{Y})$*. On the algebraic tensor product* $\mathsf{X} \odot_A \mathsf{Y}$ *balanced over* A[2]*, define the* B*-valued sesquilinear form by the formula*

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle y_1, \phi(\langle x_1, x_2\rangle_A) y_2\rangle_B.$$

Then $\langle\cdot,\cdot\rangle$ *satisfies all the conditions of the inner product for a Hilbert* C^**-module except that there may be vectors of "length" zero. Dividing out by these, one obtains a pre-Hilbert* C^**-module over* B *whose completion is denoted by* $\mathsf{X} \otimes_A \mathsf{Y}$ *and is called the* internal tensor product *of* X *and* Y.

The proof of this proposition is also straightforward and so will be omitted. Quite often, the emphasis on ϕ will be dropped and we will write ay for $\phi(a)y$. One such instance occurs when we discuss strong Morita equivalence and identify one C^*-algebra with the compact operators on a Hilbert C^*-module over the other. The following proposition justifies introducing the dual module and internal tensor product, proving that strong Morita equivalence is an equivalence relation.

COROLLARY 5.16. *If* ${}_A\mathsf{X}_B$ *is an equivalence bimodule between* C^**-algebras* A *and* B*, and if* ${}_B\mathsf{Y}_C$ *is one between* B *and the* C^**-algebra* C*, then* $\mathsf{X} \otimes_B \mathsf{Y}$ *is an equivalence bimodule between* A *and* C*. Also,* $\tilde{\mathsf{X}}$ *is an equivalence bimodule between* B *and* A*, with* $\tilde{\mathsf{X}} \otimes_A \mathsf{X}$ *naturally isomorphic to* A *as an* A*-*A *equivalence and* $\mathsf{X} \otimes \tilde{\mathsf{X}}$ *naturally isomorphic to* B *as a* B*-*B*-equivalence. Thus, strong Morita equivalence is an equivalence relation.*

THEOREM 5.17. [**28**, Theorem 1.1] *Let* X *be an* A*-*B *equivalence bimodule and let* C *be the collection of* 2×2 *matrices*

$$\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}, \tag{1.1}$$

where $a \in A$, $x \in \mathsf{X}$, $\tilde{y} \in \tilde{\mathsf{X}}$, $b \in B$*. With respect to matrix addition and product defined by the formula*

$$\begin{pmatrix} a_1 & x_1 \\ \tilde{y}_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ \tilde{y}_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + {}_A\langle x_1, y_2\rangle & a_1x_2 + x_1b_2 \\ \tilde{y}_1a_2 + b_1\tilde{y}_2 & \langle y_1, x_2\rangle_B + b_1b_2 \end{pmatrix},$$

C *becomes an algebra that acts in an obvious way on* $A \oplus \mathsf{X}$ *as bounded operators. If* C *is given the operator norm and involution,*

$$\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}^* = \begin{pmatrix} a^* & y \\ \tilde{x} & b^* \end{pmatrix},$$

then C *becomes a* C^**-algebra. In fact,* C *is isomorphic to* $\mathbb{K}(A \oplus \mathsf{X})$*. Further, the matrices*

$$p = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix},$$

[2]This is the quotient of $\mathsf{X} \odot \mathsf{Y}$ by the subspace generated by the tensors $\{xa \otimes y - x \otimes \phi(a)y \mid a \in A,\ x \otimes y \in \mathsf{X} \odot \mathsf{Y}\}$.

where 1_A *and* 1_B *denote the identities of the multiplier algebras of* A *and* B, *are projections in the multiplier algebra of* C *with the properties that* $p + q = 1_C$, $A \simeq pCp$, $B \simeq qCq$, *and the ideals* CpC *and* CqC *are dense in* C. *(Thus,* A *and* B *are isomorphic to what are called* complementary full corners of C.*)*

Conversely, if C *is a* C^**-algebra and if* p *and* q *are projections in the multiplier algebra of* C *such that* $A := pCp$ *and* $B := qCq$ *are complementary full corners, then* $\mathsf{X} := pCq$ *is an* A-B*-equivalence with inner products given by the formulae:* ${}_A\langle x, y\rangle := xy^*$, *and* $\langle x, y\rangle_B = xy^*$, $x, y \in \mathsf{X}$.

We do not need the proof here and refer the reader to the original source [**28**] or to [**156**] for details. Among other things, this result makes it clear that an equivalence bimodule is an operator space. The algebra (1.1) is called the *linking algebra* for A and B determined by X. We state one more result, to which we shall refer later. Again, we omit the proof, referring to [**28**] or to [**156**] for details. The result does not require that the C^*-algebras in question be separable; a weakened form is sufficient.

THEOREM 5.18. [**28**, Theorem 1.2] *If the* C^**-algebras* A *and* B *are* stably isomorphic *in the sense that* $A \otimes K \simeq B \otimes K$, *then* A *and* B *are strongly Morita equivalent. Conversely, if* A *and* B *have countable approximate identities*[3], *and if* A *and* B *are strongly Morita equivalent, then* A *and* B *are stably isomorphic.*

Strongly Morita equivalent C^*-algebras have "isomorphic representation theories", i.e., isomorphic categories of Hilbert modules. This will be made precise through the following definition.

DEFINITION 5.19. *A (left)* Hilbert *module over a* C^**-algebra* A, *or a (left)* Hilbert A*-module, is a Hilbert space* H *that is a left* A *module in the algebraic sense such that* $(a\xi, \eta) = (\xi, a^*\eta)$, $\xi, \eta \in H$, $a \in A$. *Equivalently,* H *is a Hilbert module iff there is a* C^**-representation* $\pi : A \to B(H)$ *such that* $a\xi = \pi(a)\xi$. *An* isomorphism *between Hilbert modules is simply a Hilbert space isomorphism intertwining the two representations.*

One defines right Hilbert modules analogously. However, in these notes, we shall have no significant use for right Hilbert modules, so unless specified to the contrary all Hilbert modules over C^*-algebras will be *left* Hilbert modules. Of course the terminology 'Hilbert module' and 'Hilbert C^*-module' may lead to confusion; care must be exercised when using them. There are important distinctions between the notions, yet the terminology does not reflect them. Unfortunately, both terms now seem fixed in the literature.

The "Equivalently" assertion in the definition requires the use of the closed graph theorem. However, when dealing with pre-C^*-algebras acting on pre-Hilbert spaces, a situation we shall face from time to time, it is necessary to add the assumption: $\|a\xi\| \le \|a\|\|\xi\|$. Then the completion of the pre-Hilbert space will be a Hilbert module over the completion of the pre-C^*-algebra.

PROPOSITION AND DEFINITION 5.20. *Let* X *be a Hilbert* C^**-module over* B, *let* A *be a sub-*C^**-algebra of* $\mathbb{B}(\mathsf{X})$, *and let* H *be a Hilbert* B*-module. On the algebraic tensor product* $\mathsf{X} \odot_B H$, *balanced over* B, *define the* $\mathbb{C}$*-valued sesquilinear form by*

[3] This assumption is equivalent to the hypothesis in [**28**, Theorem 1.2] that the C^*-algebras have strictly positive elements; see [**147**, Theorem 3.10.5].

the formula

$$(x_1 \otimes \xi_1, x_2 \otimes \xi_2) = (\xi_1, \langle x_1, x_2\rangle_B \xi_2)_H.$$

Then $(\cdot,\cdot)$ *satisfies all the conditions of the inner product for a Hilbert* C^**-module except that there may be vectors of "length" zero. Dividing out by these, one obtains a pre-Hilbert module over* A *whose completion is denoted by* ${}^A H$ *or* ${}^A_{\mathsf{X}} H$ *and is called the* Hilbert A-module induced by H via X. *If the associated representation of* B *on* H *is* π*, then the associated representation of* A *on* ${}^A H$ *is called the* representation induced by π *and is denoted by* $\operatorname{Ind}\pi$ *or* $\operatorname{Ind}^H_{\mathsf{X}}\pi$.

PROOF. The first part of this proposition is really just a special case of Proposition 5.15, with H playing the role of Y. The algebra B in that proposition is replaced by $\mathbb{C}$ in this one. The only thing that needs to be checked, really, is that A acts by bounded operators on ${}^A H$. This is a straightforward estimate using the fact that $\langle Tx, Tx\rangle_B \le \|T\|^2\langle x, x\rangle_B$ for all $x \in X$ and all $T \in \mathbb{B}(\mathsf{X})$. □

The Hilbert modules over a C^*-algebra B, say, form the objects of a category. For the morphisms, we take *continuous*, i.e. bounded, linear module maps. That is, if H_1 and H_2 are two Hilbert modules over B, then a module map is simply a bounded linear transformation $T : H_1 \to H_2$ such that $Tb\xi = bT\xi$, for all $\xi \in H$, and all $b \in B$. Expressed in terms of representations, module maps are simply (continuous) *intertwining* maps. The following proposition expresses the fact that the process of inducing Hilbert modules implements a *functor* between categories of Hilbert modules. The proof is straightforward.

PROPOSITION 5.21. *Let* X *be a Hilbert* C^**-module over the* C^**-algebra* B *and let* A *be a sub-*C^**-algebra of* $\mathbb{B}(\mathsf{X})$*. If* H_1 *and* H_2 *are two Hilbert modules over* B *and if* $T : H_1 \to H_2$ *is a* B*-module map, then* $I \otimes T$*, defined initially on* $\mathsf{X} \odot_B H_1$ *extends to a bounded linear map, also denoted by* $I \otimes T$*, from* ${}^A H_1$ *to* ${}^A H_2$ *that is an* A*-module map. Further, we have* $I \otimes (T_1 \oplus T_2) = (I \otimes T_1) \oplus (I \otimes T_2)$*,* $I \otimes TS = (I \otimes T)(I \otimes S)$*, and* $(I \otimes T)^* = I \otimes T^*$*, under the obvious hypotheses.*

This result leads immediately to the following Theorem of Rieffel that shows that strongly Morita equivalent C^*-algebras have isomorphic categories of Hilbert modules.

THEOREM 5.22. [**177**, Theorem 6.23] *If* X *is an* A*-*B*-equivalence bimodule, and if* H *is a left* B*-module, then as* B*-modules* H *and* ${}^B_{\tilde{\mathsf{X}}}\left({}^A_{\mathsf{X}} H\right)$ *are naturally isomorphic. Further, the correspondence*

$$H \to {}^A_{\mathsf{X}} H$$

determines a natural equivalence between the category of Hilbert B*-modules and the category of Hilbert* A*-modules that preserves irreducibility. The inverse is* $H \to {}^B_{\tilde{\mathsf{X}}} H$.

It is very important in the theory to know when a representation of, or a Hilbert module over, a C^*-algebra is induced. The following theorem gives a criterion which turns out to be enormously useful, even though at first glance it may seem somewhat difficult to apply. It is Rieffel's generalization of Mackey's imprimitivity theorem and it justifies the use of the term 'imprimitivity algebra' for $\mathbb{K}(\mathsf{X})$.

THEOREM 5.23. [**177**, Theorem 6.29] *Let* X *be a Hilbert* C^**-module over the* C^**-algebra* B *and let* A *be a subalgebra of* $\mathbb{B}(\mathsf{X})$*. Then a Hilbert* A*-module* K *is* *(isomorphic to) a Hilbert module induced from a Hilbert module over B via X if and only if K may be given the structure of a $\mathbb{K}(\mathsf{X})$-module in such a way that*

$$a(k\xi) = (ak)\xi$$

for all $a \in A$, $k \in \mathbb{K}(\mathsf{X})$, and $\xi \in K$.

We close this section with a somewhat technical lemma that will, nevertheless, prove quite useful to us. It is due to Phil Green (see Lemma 2 of [**82**] and its proof).

LEMMA 5.24. *Let A_0 and B_0 be pre-C^*-algebras with completions A and B. Let X_0 be an A_0-B_0-bimodule that is endowed with sesquilinear maps ${}_{A_0}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{B_0}$ satisfying conditions 1) and 2) of Definition 5.1 and condition 2) of Definition 5.10. Suppose, too, that A_0 (resp. B_0) contains an approximate identity $\{e_\alpha\}$ (resp. $\{f_\beta\}$) (not necessarily normalized) such that each e_α (resp. f_β) is a finite sum of the form $\sum {}_{A_0}\langle x_i, x_i\rangle$, $x_i \in \mathsf{X}_0$ (resp. $\sum \langle y_j, y_j\rangle_{B_0}$, $y_j \in \mathsf{X}_0$) and such that $\langle e_\alpha\eta, \eta\rangle_{B_0} \to \langle \eta,\eta\rangle_{B_0}$ while ${}_{A_0}\langle \xi f_\beta, \xi\rangle \to {}_{A_0}\langle\xi,\xi\rangle$ for all $\xi,\eta \in \mathsf{X}_0$. Then ${}_{A_0}\langle\xi,\xi\rangle \geq 0$ and $\langle\eta,\eta\rangle_{B_0} \geq 0$, for all $\xi,\eta\in\mathsf{X}_0$, and the ideals ${}_{A_0}\langle\mathsf{X}_0,\mathsf{X}_0\rangle$ and $\langle\mathsf{X}_0,\mathsf{X}_0\rangle_{B_0}$ are dense in A_0 and B_0, respectively. Thus, if ${}_{A_0}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{B_0}$ are non-singular and if the boundedness conditions of Definition 5.10 are satisfied, then the completion X of X_0 becomes an A-B-equivalence bimodule.*

PROOF. We have $\langle e_\alpha\eta,\eta\rangle_{B_0} = \sum\langle {}_{A_0}\langle x_i, x_i\rangle\eta,\eta\rangle_{B_0}$

$$\begin{aligned}
&= \sum\langle x_i\langle x_i,\eta\rangle_{B_0},\eta\rangle_{B_0}\\
&= \sum\langle \eta, x_i\langle x_i,\eta\rangle_{B_0}\rangle_{B_0}^*\\
&= \sum(\langle\eta,x_i\rangle_{B_0}\langle x_i,\eta\rangle_{B_0})^*\\
&= \sum(\langle x_i,\eta\rangle_{B_0})^*(\langle\eta,x_i\rangle_{B_0})^*\\
&= \sum(\langle x_i,\eta\rangle_{B_0})^*\langle x_i,\eta\rangle_{B_0} \geq 0.
\end{aligned}$$

Since $\langle e_\alpha\eta,\eta\rangle_{B_0} \to \langle\eta,\eta\rangle_{B_0}$, by hypothesis, we conclude that $\langle\eta,\eta\rangle_{B_0}\geq 0$, for all $\eta\in\mathsf{X}_0$. In like fashion, one shows that ${}_{A_0}\langle\xi,\xi\rangle \geq 0$ in A_0. To see that ${}_{A_0}\langle\mathsf{X}_0,\mathsf{X}_0\rangle$ is dense in A_0, simply note that for $a\in A_0$, $ae_\alpha\to a$, but $ae_\alpha = \sum {}_{A_0}\langle ax_i, x_i\rangle$ lies in ${}_{A_0}\langle\mathsf{X}_0,\mathsf{X}_0\rangle$. The density of $\langle\mathsf{X}_0,\mathsf{X}_0\rangle_{B_0}$ in B_0 is proved similarly. □

2. Equivalence of Groupoids

We return to our basic topological assumptions made throughout these notes. Groupoids will be locally compact, Hausdorff, and separable. Likewise, all general topological spaces involved will be assumed to be locally compact, Hausdorff, and separable.

In Definition 2.16, we defined the set theoretic or algebraic notion of equivalent groupoids. In the topological setting, we require more than just algebraic conditions. First, recall (Remark 2.32) that for a continuous left action of a groupoid G on a topological space X the map $r : X \to G^{(0)}$ must be continuous and open. Likewise, for right actions, s must be continuous and open. It is worthwhile to point out that the assumption that r or s be open for actions of groupoids is one that is not universally adopted in the literature. However, for our purposes, it is essential. Of course G-actions are groupoid counterparts to modules. The analogue of a Hilbert C^*-module is a principal G-space, a notion that we now develop.

DEFINITION 5.25. *Suppose the groupoid G acts continuously on space X. Then we say the action is* proper *if and only if the map Φ from $G * X$ to $X \times X$ defined by the formula $\Phi(\gamma, x) = (\gamma x, x)$ is proper in the usual sense, i.e., for each compact subset $K \subseteq X \times X$, $\Phi^{-1}(K)$ is compact in $G * X$.*

Properness for right actions has the obvious parallel definition. Observe that if a groupoid acts properly on a space, then the isotropy groups of points in the space are all compact. Much more is true, as the following proposition from [**140**] shows.

PROPOSITION 5.26. *Let G be a groupoid acting on a space X to the left, say, and assume that the action is free (i.e., assume that if $\gamma x = x$ then $\gamma = r(x)$.) Then, with the notation of Definition 5.25, the following assertions are equivalent:*

1. *The action is proper, i.e., Φ is a proper map.*
2. *Φ is a closed map.*
3. *Φ is a homeomorphism of $G * X$ onto a closed subset of $X \times X$ (with the product topology).*
4. *Given a compact subset $K \subseteq X$, the set $G(K) := \{\gamma \in G \mid \gamma K \cap K \neq \emptyset\}$ is compact in G.*
5. *Given a compact set $K \subseteq X$, $G(K)$ is relatively compact in G.*

PROOF. The equivalence of the first three assertions follows from [**25**, I.10.1 Proposition 2]. If Φ is proper and if $K \subseteq X$ is compact, then $\Phi^{-1}(K \times K)$ is compact in $G*X$. But then $G(K)$, which is the projection of $\Phi^{-1}(K \times K)$ onto the first factor, G, is compact. Thus each of the first three assertions implies the fourth. Obviously, the fourth implies the fifth. If the fifth is satisfied, then for each compact $K \subseteq X$, $G(K)$ is relatively compact and so, therefore, is $\Phi^{-1}(K \times K) \subseteq K * G(K)$. Since $\Phi^{-1}(K \times K)$ is closed, we conclude that $\Phi^{-1}(K \times K)$ is compact. This is enough to conclude that Φ is proper. $\square$

Compact sets K in X with the property that $G(K)$ is relatively compact are called *wandering*. Thus, the proposition says that the action is proper if and only if all compact sets are wandering. The proof of the following proposition is taken from [**140**, Lemma 2.1] and [**80**, Theorem 14].

PROPOSITION 5.27. *Let G act continuously on a space X, on the left, say. Then the quotient map π from X to $G\backslash X$ is open. If the action is proper, then the quotient space $G\backslash X$ is Hausdorff.*

PROOF. Suppose U is open in X. To show that $\pi(U)$ is open, it suffices to show that $G \cdot U = \{\gamma x \mid (\gamma, x) \in G * U\}$ is open in X. If $x_i \to \gamma x \in G \cdot U$, then $r(x_i) \to r(\gamma \cdot x) = r(\gamma)$. Since r is open, we may pass to a subnet and assume that there are γ_i in G converging to γ such that $r(\gamma_i) = r(x_i)$. Then $\gamma_i^{-1} x_i \to x$ and so $\gamma_i^{-1} x_i$ is eventually in U and therefore $x_i = \gamma_i(\gamma_i^{-1} x_i)$ is eventually in $G \cdot U$. Thus $G \cdot U$ is open. If $G\backslash X$ were not Hausdorff, we could find a net $\{[w_\alpha]\}$ in $G\backslash X$ converging to two distinct points $[y_1]$ and $[y_2]$, where we write $[x]$ for $\pi(x)$. Since π is open, we may find two nets $\{x_\alpha\}$ and $\{x'_\alpha\}$, with $[x_\alpha] = [x'_\alpha] = [w_\alpha]$ such that $x_\alpha \to y_1$ and $x'_\alpha \to y_2$. Since $[x_\alpha] = [x'_\alpha]$, we may write $x'_\alpha = \gamma_\alpha x_\alpha$ for suitable $\gamma_\alpha \in G$. Choose a compact neighborhood U_i of y_i, $i = 1, 2$. Since the action is proper, the set $U_1 \cup U_2$ is wandering, by Proposition 5.26. Consequently, the γ_α all lie in some compact subset of G for sufficiently large α. Passing to a subnet, if necessary, we may assume that the γ_α converge to some γ. Then by joint continuity of the action, $x_\alpha \to y_1$, and $x'_\alpha = \gamma_\alpha x_\alpha \to \gamma y_1$. Since $x'_\alpha \to y_2$ and X is Hausdorff, we conclude that $\gamma y_1 = y_2$, contradicting the assumption that $[y_1]$ $(= [\gamma y_1])$ is distinct from $[y_2]$. □

DEFINITION 5.28. *Let G be a groupoid acting on a space X. We call X a* principal G-space *if the action is free and proper. If X and Y are two principal G-spaces, then a* homomorphism *from X to Y is simply a continuous, equivariant map from X to Y. An* isomorphism *is an equivariant homeomorphism.*

In the literature, a principal G-space, where G is a group, is also called a *Cartan principal bundle* [**113**]. This is to distinguish the structure from what is usually called a principal G-bundle, vis., a locally trivial Cartan principal bundle. The notion of a locally trivial principal G-space makes sense in the case of groupoids and plays important roles in applications of groupoids to geometry. Such spaces arise, too, in operator algebra, as we shall see in Chapter 7. I.O.U.

Examples of principal G-spaces are easy to obtain, but there are certain difficulties with which one must come to grips in our general setting in order for the theory to function smoothly.

EXAMPLES 5.29. 1. *The simplest is the case when the groupoid G acts upon itself by either right or left translation.*

2. *More generally, let F be a closed or open subset of $G^{(0)}$ and consider the space G_F $(= s^{-1}(F))$. In either case, $G|_F$ is a locally compact groupoid. Set theoretically, $G|_F$ acts on G_F to the right and the action is certainly free, but the continuity and properness of the action are at issue. First of all, if F is open, then certainly the restriction of s to G_F is open as a map from G_F onto F and then it is easy to see that the action is continuous. Properness is also clear since the map $(\gamma, \alpha) \to (\gamma, \gamma\alpha)$, from $G_F \times G|_F$ to $G_F \times G_F$, has inverse $(\xi, \eta) \to (\xi, \xi^{-1}\eta)$. Thus, when F is open, G_F is a principal $G|_F$-space. When F is closed, the situation becomes more complicated. The restricted map $s|G_F$ need not be open, but it if it is, then again G_F becomes a right principal $G|_F$ space.*

   Now assume that r maps G_F onto $G^{(0)}$. This is the same as assuming that the smallest invariant subset of $G^{(0)}$ containing F is all of $G^{(0)}$. Then it is easy to see that G_F becomes a left principal G-space precisely when the restriction of r to G_F is open as a map from G_F to $G^{(0)}$. The problem is that this does not seem always to be the case. The transformation groupoid determine by the discrete *reals acting on $\mathbb{R}$ with the usual topology yields a transformation group groupoid such that $r|G_u$ is not open for any $u \in G^{(0)} = \mathbb{R}$. In many cases of interest, however, $r|G_F$ is open.*

3. *A particular case where $r|G_F$ is open is when G is transitive (and locally compact, Hausdorff, and second countable, as we usually assume) and $F = \{u\}$ for some unit $u \in G^{(0)}$. This is proved in Theorems 2.2A and 2.2B in* [**136**]. *Thus, in this case, for each $u \in G^{(0)}$, G_u is a left principal G-space.*

As we said at the outset of this section, the notion of principal G-space is an analogue of a Hilbert C^*-module. The groupoid analogue of the imprimitivity algebra of a Hilbert C^*-module is called the imprimitivity groupoid (of a principal G-space). We met this notion in Chapter 2 (Definition 2.15), but without any topology involved. Here we take a moment to develop the salient features. We begin with the following lemma from [**136**, page 7] that proves to be useful in a number of contexts.

LEMMA 5.30. *Suppose that X, Y, and Z are topological spaces (not necessarily locally compact, etc.) and that f and g are maps from X and Y, respectively, into Z. Let $X * Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ and give $X * Y$ the relative topology in $X \times Y$. Consider the commutative diagram*

$$\begin{array}{ccccc} & & X & & \\ & \pi_X \nearrow & & \searrow f & \\ X * Y & & & & Z\ , \\ & \pi_Y \searrow & & \nearrow g & \\ & & Y & & \end{array}$$

where π_X and π_Y are the projections of $X * Y$ onto X and Y, respectively. If f is open and g is continuous, then is π_Y is open.

PROOF. Indeed, if U and V are open in X and Y, respectively, then $\pi_Y((U \times V) \cap (X * Y)) = g^{-1}[f(U)] \cap V$. Since sets of the form $(U \times V) \cap (X * Y)$ constitute a basis for the topology on $X * Y$, the result follows. The following theorem is contained in Theorem 3.5 of [**140**]. ☐

THEOREM 5.31. *Let H be a groupoid and X a right principal H-space. Then*

1. *With respect to the quotient topology, $X *_H X^{\mathrm{op}}$ is a (locally compact, Hausdorff, second countable) groupoid.*
2. *The canonical left action of $X *_H X^{\mathrm{op}}$ on X is continuous, free, and proper, making X a left principal $X *_H X^{\mathrm{op}}$-space.*
3. *The actions of H and $X *_H X^{\mathrm{op}}$ commute, and $s : X \to X/H$ induces a homeomorphism from X/H onto the unit space of $X *_H X^{\mathrm{op}}$ while $r : X \to X *_H X^{\mathrm{op}} \backslash X$ induces a homeomorphism from $X *_H X^{\mathrm{op}} \backslash X$ onto $H^{(0)}$.*

Thus, X is an equivalence between H and $X *_H X^{\mathrm{op}}$ in the sense of the following definition.

DEFINITION 5.32. *Let G and H be groupoids. We say that a space X is a* (G, H)-equivalence *in case*

1. *X is a principal left G-space and a principal right H-space.*
2. *The actions of G and H commute.*
3. *The map $r : X \to G^{(0)}$ induces a homeomorphism between $G^{(0)}$ and X/H, i.e., $r(x) = r(y)$ if and only if there is a (necessarily unique) η such that $x\eta = y$, and the map $s : X \to H^{(0)}$ induces a homeomorphism between $H^{(0)}$ and $G\backslash X$.*

Further, we shall say that two locally compact groupoids G and H are Morita equivalent *in case there is a (G, H)-equivalence.*

Of course, the terminology presumes that being related by having an 'equivalence' is an equivalence relation among groupoids. This is indeed the case and easily proved, so we shall not pause to prove it here.

PROOF OF THEOREM 5.31.

Evidently, since X is a locally compact principal H-space, so is $X * X^{\mathrm{op}}$, with the relative topology. By Proposition 5.27, $X *_H X^{\mathrm{op}}$ is a locally compact Hausdorff space with the quotient topology. Inversion ($[x,y] \to [y,x]$) is certainly continuous. To see that multiplication is continuous, we use Lemma 5.30: Consider the commutative diagram:

$$\begin{array}{ccc} X * X^{\mathrm{op}} & \stackrel{p_1}{\to} & X \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ X *_H X^{\mathrm{op}} & \stackrel{r}{\to} & X/H \end{array},$$

where p_1 denotes the projection onto the first factor, and π and $\tilde{\pi}$ are the quotient maps. The maps π and $\tilde{\pi}$ are continuous and open by Lemma 5.27. Also, since $X * X^{\mathrm{op}}$ is a fibred product over continuous open maps s and r^{op}, p_1 is open by the open factor lemma, Lemma 5.30. Consequently, r is continuous and open by a straightforward diagram chase. Of course the source map on $X *_H X^{\mathrm{op}}$ is continuous and open since it is the composition of r and inversion.

To see that multiplication is continuous, consider the following diagram

$$\begin{array}{ccc} X * X^{\mathrm{op}} * X & \stackrel{\tilde{m}}{\to} & X * X^{\mathrm{op}} \\ \tilde{\rho}_1 \downarrow & & \downarrow \tilde{\rho} \\ (X *_H X^{\mathrm{op}}) * (X *_H X^{\mathrm{op}}) & \stackrel{m}{\to} & X *_H X^{\mathrm{op}} \end{array}$$

where $\tilde{\rho}$ is the quotient map, $\tilde{m}(x_1, x_2, x_3) = (x_1, x_3)$, m is multiplication on $X *_H X^{\mathrm{op}}$, and $\tilde{\rho}_1(x_1, x_2, x_3) = ([x_1, x_2], [x_2, x_3])$. Since $\tilde{\rho}$ and $\tilde{m}$ are continuous, to show that m is continuous, it suffices to prove that the range of $\tilde{\rho}_1$ is all of $(X *_H X^{\mathrm{op}}) * (X *_H X^{\mathrm{op}})$ and that $\tilde{\rho}_1$ is open. For the first assertion, observe that $([x_1, x_2], [x_2', x_3]) \in (X *_H X^{\mathrm{op}}) * (X *_H X^{\mathrm{op}})$ iff $[x_2] = [x_2']$ and this happens if only if there is an $\eta \in H$ such that $x_2\eta = x_2'$. In this event, $[x_2', x_3] = [x_2\eta, x_3] = [x_2, x_3\eta]$. Therefore, $\tilde{\rho}_1(x_1, x_2, x_3\eta) = ([x_1, x_2], [x_2', x_3])$. For the second assertion, note that $\tilde{\rho}$ is open by Lemma 5.27. Also, note that if $x_\alpha \to x$, and $[x_\alpha, y_\alpha] \to [x, y]$, then $y_\alpha \to y$. (To see this, it suffices to show that every subnet of $\{y_\alpha\}$ has a subnet converging to y. So assume, after relabeling, that $\{y_\alpha\}$ is a given subnet of the original net of y's. The openness of $\tilde{\rho}$ implies that we may pass to a subnet of $\{y_\alpha\}$, if necessary, which we continue to write $\{y_\alpha\}$, and find $\eta_\alpha \in H$ such that $(x_\alpha\eta_\alpha, \eta_\alpha^{-1}y_\alpha) \to (x, y)$. Thus, in particular, $x_\alpha\eta_\alpha \to x$. The properness of the H-action on X, and the fact that $x_\alpha \to x$, imply that $\eta_\alpha \to s(x) = r^{\mathrm{op}}(y)$. Therefore, $y_\alpha = \eta_\alpha(\eta_\alpha^{-1}y_\alpha) \to r^{\mathrm{op}}(y) \cdot y = y$.) So, to show that $\tilde{\rho}_1$ is open, it suffices to show that if a net $\{([x_1^\alpha, x_2^\alpha], [x_2^\alpha, x_3^\alpha])\}$ converges to $([x_1, x_2], [x_2, x_3])$ in $(X *_H X^{\mathrm{op}}) * (X *_H X^{\mathrm{op}})$, then a subnet of $\{(x_1^\alpha, x_2^\alpha, x_3^\alpha)\}$ converges in $X * X^{\mathrm{op}} * X$. Since $\{[x_1^\alpha, x_2^\alpha]\}$ converges to $[x_1, x_2]$, we may pass to a subnet, if necessary, and find $\eta_\alpha \in H$, such that $\{x_1^\alpha \cdot \eta_\alpha\}$ and $\{\eta_\alpha^{-1} \cdot x_2^\alpha\}$ converge. We may therefore replace $x_1^\alpha \cdot \eta_\alpha$ by x_1^α and replace $\eta_\alpha^{-1} \cdot x_2^\alpha$ by x_2^α and assume that $\{x_1^\alpha\}$ and $\{x_2^\alpha\}$ converge. But then the assertion that we just proved shows that necessarily $\{x_3^\alpha\}$ converges, too, proving that $\tilde{\rho}_1$ is open. This completes the proof of the first assertion of Theorem 5.31.

For the second, recall that the map r from X to $(X *_H X^{\mathrm{op}})^{(0)} = X/H$ in the definition of the left action of $X *_H X^{\mathrm{op}}$ on X is just the quotient map π. So $(X *_H X^{\mathrm{op}}) * X = \{([x_1, x_2], x_3) \mid [x_2] = [x_3]\}$, and given $([x_1, x_2], x_3)$ in this space, there is an $\eta \in H$ (which is unique, by the free action of H on X) such that $x_2 = \eta x_3$. The action of $[x_1, x_2]$ on x_3 is defined by the equation $[x_1, x_2] \cdot x_3 = [x_1\eta, x_3] \cdot x_3 = x_1\eta$. That this determines a well defined, free action of $X *_H X^{\mathrm{op}}$

on X is now easily checked. To see that it is continuous, first consider the map $\Phi : X * H \to X \times X$ defined by the equation $\Phi(x, \eta) = (x, x \cdot \eta)$. Then the fact that X is a principal H-space means that Φ is a homeomorphism onto its range, $X *_r X$ (Lemma 5.26). It follows that the map $\delta : X *_r X \to H$, defined by the formula $\delta(x, x \cdot \eta) = \eta$, is continuous, since it is the composition of Φ^{-1} followed by the projection onto the second factor. (In the literature, δ is sometimes called the *division map* associated with the principal H-space [**113**].) The continuity of the $X *_H X^{\mathrm{op}}$ action follows from analyzing the commutative diagram

$$\begin{array}{cccccc} & X * X^{\mathrm{op}} * X & \xrightarrow{i\times\delta} & X * H & \\ \tilde{\rho}\times i & \downarrow & & \downarrow & (x,\eta)\mapsto x\cdot\eta \\ & X *_H X^{\mathrm{op}} * X & \longrightarrow & X & \end{array}$$

where the bottom horizontal arrow represents the action $([x_1, x_2], x_3) \mapsto [x_1, x_2] \cdot x_3$. The right vertical map is continuous by assumption and we just showed that $i \times \delta$ is continuous. Since $\tilde{\rho}$ is open by Lemma 5.27, so is $\tilde{\rho} \times i$. Thus, the continuity of the lower horizontal map, i.e. the $X *_H X^{\mathrm{op}}$ action, follows from the obvious chase around the diagram. Since the $X *_H X^{\mathrm{op}}$ action is free, to show that it is proper, we need only show that the map $\Psi : X *_H X^{\mathrm{op}} * X \to X \times X$, defined by the formula $\Psi(\gamma, x) = (x, \gamma x)$, is closed. So suppose $\{\gamma_\alpha\}$ and $\{x_\alpha\}$ are nets such that $x_\alpha \to x$ and such that $\gamma_\alpha \cdot x_\alpha \to y$ in X. By definition of the $X *_H X^{\mathrm{op}}$ action, we may write each γ_α as $[y_\alpha, x_\alpha]$, with $[y_\alpha, x_\alpha] \cdot x_\alpha = y_\alpha$. By hypothesis, $y_\alpha \to y$ and $x_\alpha \to x$. Therefore, $[y_\alpha, x_\alpha] \to [y, x]$, and $y = [y, x] \cdot x$. This shows that Ψ is closed and completes the proof of the second assertion in Theorem 5.31. Since the other assertions of the theorem are now easily verified, we conclude the proof here. □

It is instructive to review Examples 5.29 in the light of Theorem 5.31 and Definition 5.32.

EXAMPLES 5.33.
1. *A groupoid is always equivalent to itself via itself. That is, if G is a locally compact groupoid, then G acts to the left and right on G by translation and it is easy to see that G becomes a G-G-equivalence.*
2. *It is possible and useful to think of equivalence as generalizing the notion of isomorphism. Specifically, if G and H are locally compact groupoids and if $\varphi : H \to G$ is a homeomorphic isomorphism, then G becomes a G-H-equivalence by letting G act on G via left translation and by letting H act on the right of G via the formula: $\gamma \cdot \eta := \gamma\varphi(\eta)$.*
3. *If G is a transitive groupoid (locally compact, Hausdorff, and second countable, as always), if $u \in G^{(0)}$ is any unit, and H is the isotropy group of u, then G_u is a G-H-equivalence. This is proved as Example 2.2 in* [**136**].[4]
4. *As a particular case of the preceding example, suppose that X is a locally compact Hausdorff space that is connected, locally arcwise connected, and semilocally simply connected, then its fundamental groupoid is transitive and equivalent, in the sense of Definition 5.32 to the fundamental group based at any point. One may use the universal covering space of X to implement the equivalence. For details of this, see* [**170**, p.133 ff.]. *One does not need to assume that X is second countable here because the groupoid is locally trivial in the sense that for each unit $u \in G^{(0)}$ the map $r|G_u$ has local continuous sections. (See* [**200**] *and* [**113**] *for discussions about locally trivial groupoids.)*

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[4] It may be helpful to note that the proof of Example 2.2 in [**136**] uses Proposition 2.40 without comment.

5. *If R is an open equivalence relation on a locally compact Hausdorff space X such that, as a subset of $X \times X$, R is closed, then X implements an equivalence between R, as a groupoid, and the locally compact space X/R as a cotrivial groupoid.*
6. *Suppose that H and K are groups acting freely and properly on a locally compact Hausdorff space X and assume the actions commute. Say K acts on the left and H acts on the right of X. Then by Lemma 5.27 the quotient spaces $K\backslash X$ and X/H are locally compact and Hausdorff and the quotient maps are open. The action of K passes to X/H and the action of H passes to $K\backslash X$. The space X, then, becomes an equivalence between the transformation group groupoids $K \times X/H$ and $K\backslash X \times H$. This observation was first made by Rieffel in* [**178**].
7. *If G is a locally compact groupoid and if $F \subseteq G^{(0)}$ is a closed subset that meets each orbit in $G^{(0)}$, then if the restrictions of r and s to G_F are open, then G_F is a G-$G|_F$-equivalence. This situation occurs in the theory of foliations. If G is the holonomy groupoid of a foliation and if F is a (not-necessarily-connected) transverse submanifold that meets each leaf, then the restrictions of r and s to G_F are open and G_F is a G-$G|_F$-equivalence. (See* [**37**] *and* [**93**].*) A particular instance of this may be found in Example 1.1. The transformation group groupoid G determined by $\mathbb{R}$ acting on the 2-torus, as described there, is the holonomy groupoid of the foliation of the 2-torus by lines of slope α. The unit space of the groupoid is $\mathbb{T}^2$. The circle $F := \{[0, \theta] \mid \theta \in \mathbb{R}\}$ is a transversal and it is a simple matter to check that the restrictions of r and s to G_F are open. The groupoid $G|_F$ is the groupoid determined by the action of $\mathbb{Z}$ acting on $\mathbb{T}$ through translation by α. (See Example 4.26.)*

REMARK 5.34. *In connection with the last example, it should be noted that there are situations where one may "cut down" to a transversal that is neither open nor closed in order to build Morita equivalent groupoids. For a discussion of some of the things that can happen here, see* Get references from Ian and Jack.

REMARK 5.35. *Suppose that G and H are groupoids of the kind we have been considering and suppose that X is a (G, H)-equivalence. Then we can build a new groupoid, denoted L, that is the disjoint union of G, H, X, and X^{op} arranged formally as a matrix*

$$L = \begin{pmatrix} H & X^{\mathrm{op}} \\ X & G \end{pmatrix}.$$

The notation is to indicate the obvious multiplication rules, where we have to remember that we identify H with $X *_G X^{\mathrm{op}}$ and G with $X^{\mathrm{op}} *_H X$, so that for $x \in X$, and $y \in X^{\mathrm{op}}$, $[x, y]$ is viewed as an element in H and $[y, x]$ is viewed as an element of G. When this is done, it is easy to check that L is a groupoid with unit space $H^{(0)} \cup G^{(0)}$. In fact, giving L the natural topology, making H, G, X, and X^{op} all open and restricting to the original topologies on each of the pieces, we see that L is a locally compact groupoid with the property that $H^{(0)}$ and $G^{(0)}$ are complementary closed and open sets, each meeting every orbit in $L^{(0)}$, whose union is $L^{(0)}$. Further, $L|_{G^{(0)}}$ is naturally isomorphic and homeomorphic to G and $L|_{H^{(0)}}$ is naturally isomorphic and homeomorphic to H. The groupoid L is called the linking groupoid of the equivalence X. *It is the groupoid analogue of the linking algebra of a strong*

Morita equivalence beteween C^*-algebras as described in Theorem 5.17. Conversely, if L is locally compact groupoid and if U and V are complementary closed and open subsets of $L^{(0)}$ that meet each orbit, then the restrictions of r and s to L^U and to L^V are continuous and open. It follows, then, that the three groupoids L, $L|_U$, and $L|_V$ are all Morita equivalent. In fact, L_V^U is an $L|_U$-$L|_V$-equivalence and L is homeomorphic and isomorphic to the linking groupoid determined by $L|_U$, $L|_V$, and L_V^U. Thus we see that Morita equivalence of groupoids can be fully carried out in the context of linking groupoids. (For completeness and accuracy, we should note that no mention has been made of Haar systems here. These will be incorporated in Section 5.) This approach Morita equivalence is taken in Kumjian's work listed in the bibliography. See [**102**], *in particular.*

Show here how Kumjian's slant on equivalence shows that equivalent measured groupoids are similar. Refer back to Proposition 2.24.

3. Equivalence and Strong Morita Equivalence

Throughout this section, G and H will be locally compact groupoids of the kind we have been considering, with Haar systems $\{\lambda^u\}_{u\in G^{(0)}}$ and $\{\beta^u\}_{u\in G^{(0)}}$, respectively. We shall write A_c for $C_c(G,\lambda)$ and B_c for $C_c(H,\beta)$. These are pre-C^*-algebras and we write A and B, respectively, for their completions. Further, X will be a G-H-equivalence. The space $C_c(X)$ becomes an A_c–B_c-bimodule in a natural way. In order to define this structure, we require

LEMMA 5.36. [**136**, Lemma 2.9] *Let Z be a (left) G-space.*

a) *If $F\in C_c(G\times Z)$, then the equation*

$$\varphi(u,z) := \int_G F(\gamma,z)\,d\lambda^u(\gamma)$$

defines an element in $C_c(G^{(0)}\times Z)$.

b) *If Z is a principal left G-space and if $f\in C_c(Z)$, then the equation*

$$\tilde\lambda(f)([z]) := \int_G f(\gamma^{-1}\cdot z)\,d\lambda^{r(z)}(\gamma)$$

defines an element of $C_c(G\backslash Z)$ (where, recall, $r: Z\to G^{(0)}$ denotes the map that comes as part of the definition of left G-space and where $[z]$ denotes the orbit of $z\in Z$.)

Similar assertions hold for right actions of groupoids.

REMARK 5.37. *The map $f\to\tilde\lambda(f)$ is, in fact, a surjection from $C_c(Z)$ onto $C_c(G\backslash Z)$. This could be proved here, but we will not need the fact until the next section, when we prove Proposition 5.39. See Lemma 5.46.*

PROOF. For part a), it suffices to consider functions F of the form $F(\gamma,z) = f(\gamma)g(z)$, with $f\in C_c(G)$ and $g\in C_c(Z)$. The assertion, then, is immediate from the properties that $\{\lambda^u\}_{u\in G^{(0)}}$ has as a Haar system. For part b), view $\tilde\lambda(f)$ as a function on Z that is constant on orbits. (This is possible, since $\{\lambda^u\}_{u\in G^{(0)}}$ is left invariant.) Since the statement about supports is straightforward, we show that $\tilde\lambda(f)$ is continuous at any prescribed $z\in Z$. From the properness of the G action (see part 5) of Proposition 5.26), there is an $F\in C_c(G\times Z)$ such that $F(\gamma,z) := f(\gamma^{-1}z)$ near z. The result now follows from part a). ∎

To define the A_c-B_c-bimodule structure on $C_c(X)$, let $f \in A_c$, $g \in B_c$, and $\varphi \in C_c(X)$. Then $f \cdot \varphi$ and $\varphi \cdot g$ are defined by the formulae:

$$f \cdot \varphi(x) := \int_G f(\gamma)\varphi(\gamma^{-1}x)\, d\lambda^{r(x)}(\gamma) \tag{3.1}$$

and

$$\varphi \cdot g(x) := \int \varphi(x\eta) g(\eta^{-1})\, d\beta^{s(x)}(\eta). \tag{3.2}$$

Part a) of Lemma 5.36 shows that $f \cdot \varphi$ and $\varphi \cdot g$ are, indeed, in $C_c(X)$. It is straightforward to check the various algebraic laws that must be satisfied to make $C_c(X)$ an A_c-B_c-bimodule and so these will be omitted.

The space $C_c(X)$ also is endowed with A_c- and B_c- sesquilinear forms, denoted ${}_A\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_B$, respectively. They are defined by the formulae:

$${}_A\langle\varphi,\psi\rangle(\gamma) = \int_H \varphi(\gamma x\eta)\overline{\psi(x\eta)}\, d\beta^{s(x)}(\eta) \tag{3.3}$$

and

$$\langle\varphi,\psi\rangle_B(\eta) = \int_G \overline{\varphi(\gamma^{-1}x)}\psi(\gamma^{-1}x\eta)\, d\lambda^{r(x)}(\gamma), \tag{3.4}$$

where in the first formula, x is any element of X such that $r(x) = s(\gamma)$, and in the second formula, x is arbitrary, subject to the constraint that $s(x) = r(\eta)$. Of course, we need to check that these expressions are well defined, i.e., independent of x, and give bona fide functions in A_c and B_c respectively. Consider x in equation 3.3. If x_1 also satisfies $r(x_1) = s(\gamma) = r(x)$, then since X is a G-H-equivalence, there is a unique η_0 such that $x_1 = x\eta_0$. Since the Haar system β is left invariant, $\int_H \varphi(\gamma x\eta)\overline{\psi(x\eta)}\, d\beta^{s(x)}(\eta) = \int_H \varphi(\gamma x_1\eta)\overline{\psi(x_1\eta)}\, d\beta^{s(x)}(\eta)$. A similar argument works for x in equation 3.4. To show that these functions are in A_c and B_c, first note that $X *_s X := \{(x,y) \in X \times X \mid s(x) = s(y)\}$ is a principal G-space. Therefore, for φ and ψ in $C_c(X)$, the function

$$\Phi(x,y) := \int \varphi(\gamma^{-1}x)\psi(\gamma^{-1}y)\, d\lambda^{r(x)}(\gamma)$$

defines an element in $C_c(G\backslash X *_s X)$ by part b) of Lemma 5.36. But also, $\langle\varphi,\psi\rangle_B(\eta) = \Phi(x, x\eta)$, which is independent of x satisfying $s(x) = r(\eta)$, as we just observed. Also, it follows from the properness of the H action that $\langle\varphi,\psi\rangle_B(\cdot)$ has compact support in H. For continuity, simply observe that if $\eta_\alpha \to \eta$ in H, then since s and r are open, we can pass to a subnet if necessary and find $x_\alpha \to x$ in X such that $s(x_a) = r(\eta_\alpha)$. But then $\langle\varphi,\psi\rangle_B(\eta_\alpha) = \Phi(x_\alpha, x_\alpha\eta_\alpha) \to \Phi(x, x\eta) = \langle\varphi,\psi\rangle_B(\eta)$. This implies that $\langle\varphi,\psi\rangle_B(\cdot)$ is continuous. A similar argument shows that ${}_A\langle\varphi,\psi\rangle(\cdot)$ is in A_c.

THEOREM 5.38. [**136**, Theorem 2.8] *If G and H are second countable, locally compact, Hausdorff groupoids with Haar systems λ and β respectively, and if X is a G-H-equivalence, then with respect to the operations defined in equations 3.1) — 3.4), $C_c(X)$ becomes a pre-Hilbert C^*-module over both $C_c(G,\lambda)$ and $C_c(H,\beta)$ in such a way that the completion of $C_c(X)$ (with respect to both inner products) is an equivalence bimodule between $C^*(G,\lambda)$ and $C^*(H,\beta)$.*

The proof rests squarely on the next proposition, which we prove in the next section. First, however, we list a number of algebraic facts whose proofs are straightforward calculations and will be omitted. It is important to note, nevertheless, that arguments similar to those used in the proof of Lemma 5.36 show that the operations are continuous in the inductive limit topology.

$$f \cdot {}_A\langle \varphi, \psi\rangle = \langle f\cdot\varphi, \psi\rangle,\quad f\in A_c,\ \varphi,\psi\in C_c(X).$$
$$\langle \varphi, \psi\rangle_B \cdot g = \langle \varphi, \psi\cdot g\rangle_B,\quad g\in B_c,\ \varphi,\psi\in C_c(X).$$
$${}_A\langle \varphi, \psi\rangle^* = {}_A\langle \psi, \varphi\rangle,$$
$$\langle \varphi, \psi\rangle_B^* = \langle \psi, \varphi\rangle_B, \text{ and}$$
$${}_A\langle \varphi, \psi\rangle \cdot \xi = \varphi\cdot\langle \psi, \xi\rangle_B,\quad \varphi,\psi,\xi\in C_c(X).$$

PROPOSITION 5.39. [**136**, Proposition 2.10] *Under the hypotheses of Theorem 5.38 there is a sequence $\{e_k\}$ in $C_c(G)$ consisting of elements of the form*

$$e_k = \sum_{i=1}^{n_k} {}_A\langle \varphi_i^k, \varphi_i^k\rangle,$$

where each φ_i^k lies in $C_c(X)$, that serves as a two-sided approximate identity for $C_c(G,\lambda)$ in the inductive limit topology and has the property that $e_k\cdot\varphi\to\varphi$ in the inductive limit topology on $C_c(X)$ for each $\varphi\in C_c(X)$. Similar statements hold for H.

PROOF OF THEOREM 5.38

The formulas preceding Proposition 5.39, the fact that the inductive limit topology is finer than the C^*-norm topology, and Proposition 5.39 combine to allow us to invoke Lemma 5.24. We conclude from that lemma that the inner products ${}_A\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_B$ are positive semidefinite and that their ranges are dense in A_c and B_c, respectively. We may clearly pass to the quotient of $C_c(X)$ by the space of functions of norm 0, calculated with respect to either inner product, and assume without loss of generality that the inner products are definite. The only thing left to verify, then, is that the actions of A_c and B_c on $C_c(X)$ are bounded. That is, we need to show that for all $f\in A_c$, $g\in B_c$, and $\varphi\in C_c(X)$, the following inequalities hold:

$$\langle f\cdot\varphi, f\cdot\varphi\rangle_B \le \|f\|_A^2\langle\varphi,\varphi\rangle_B,$$

and

$${}_A\langle \varphi\cdot g, \varphi\cdot g\rangle \le \|f\|_B^2\,{}_A\langle\varphi,\varphi\rangle.$$

We shall verify the first; the second is handled similarly.

Let ζ be any state on $B = C^*(H,\beta)$ and observe that $C_c(X)$ becomes a pre-Hilbert space with respect to the inner product

$$\langle\cdot,\cdot\rangle_\zeta := \zeta(\langle\cdot,\cdot\rangle_B).$$

Let K denote its Hausdorff completion and let K_0 denote the image of $C_c(X)$ in K. We shall view elements of K_0 simply as functions in $C_c(X)$. Define L from $B_c = C_c(G)$ into the linear transformations on K_0 by the formula $L(f)\xi = f\cdot\xi$, $\xi\in C_c(X)$. Then:

a) L is non-degenerate in the sense that the span of $\{L(f)\xi \mid f\in C_c(G),\ \xi\in K_0\}$ is dense in K. This is immediate from Proposition 5.39.

b) For each $\xi, \eta \in K_0$, the functional $L_{\xi,\eta} : C_c(G) \to \mathbb{C}$, defined by the formula $L_{\xi,\eta}(f) = \langle L(f)\xi, \eta\rangle_\zeta = \zeta(\langle f \cdot \xi, \eta\rangle_B)$ is continuous with respect to the inductive limit topology on $C_c(G)$. This is because $f \to \langle f \cdot \xi, \eta\rangle_B$ is continuous with respect to the inductive limit topologies on $C_c(G)$ and $C_c(H)$ and the inductive limit topology on $C_c(H)$ is finer than the C^*-norm topology on $C_c(H) \subseteq B$.

c) Finally, for all $f \in C_c(G)$, and all $\xi, \eta \in K_0$, $\langle \xi, L(f^*)\eta\rangle_\zeta = \langle L(f)\xi, \eta\rangle_\zeta$ because, as an easy calculation shows, $\langle \xi, f^* \cdot \eta\rangle_B = \langle f \cdot \xi, \eta\rangle_B$.

Thus, the three conditions of Renault's disintegration theorem, Theorem 3.32, are satisfied and we conclude from that result that L extends to C^*-representation of $C^*(G, \lambda)$ on K, proving that $\zeta(\langle f \cdot \varphi, f \cdot \varphi\rangle_B) \leq \|f\|_A^2 \zeta(\langle \varphi, \varphi\rangle_B)$ for all $f \in C_c(G)$ and all $\varphi \in C_c(X)$. Since the state ζ was chosen arbitrarily, the inequality $\langle f \cdot \varphi, f \cdot \varphi\rangle_B \leq \|f\|_A^2 \langle \varphi, \varphi\rangle_B$ is valid in the C^*-algebra $C^*(H, \beta)$, as was to be shown. □

Before turning to the proof of Proposition 5.39, we present an immediate corollary of Theorem 5.38.

COROLLARY 5.40. *Suppose G is a second countable, locally compact, Hausdorff groupoid and that $\lambda_1 = \{\lambda_1^u\}_{u \in G^{(0)}}$ and $\lambda_2 = \{\lambda_2^u\}_{u \in G^{(0)}}$ are two Haar systems on G. Then $C^*(G, \lambda_1)$ is strongly Morita equivalent to $C^*(G, \lambda_2)$.*

PROOF. As we noted in Example 5.33, G is equivalent to itself via G. Since Theorem 5.38 is independent of the Haar systems used, it proves this corollary. □

REMARKS 5.41. 1. *We know of no other way to show that the C^*-algebra of a groupoid is independent of the Haar system on it, up to strong Morita equivalence. Further, while in every example we know of, the C^*-algebras associated with two Haar systems are actually $*$-isomorphic, we do not have a proof that this is true in general.*

2. *One of the surprising things about Theorem 5.38 is that no mention of and there is apparently no need for any kind of measures on an equivalence between groupoids to conclude that the groupoid algebras are strongly Morita equivalent. Measures on equivalences do not entirely vanish from the picture, however, as we shall see in Section 5.*

Add a proof of Mackey's imprimitivity theorem here?

4. The Proof of Proposition 5.39

The proof of Proposition 5.39 rests on several lemmas. The first has applicability beyond the present purpose. For it, we require a definition (cf. Definition 3.4 and Remarks 3.5.)

DEFINITION 5.42. *Suppose that $\pi : X \to Y$ is a continuous open surjection between locally compact Hausdorff spaces. Suppose also that $\lambda = \{\lambda_y\}_{y \in Y}$ is a family of positive Radon measures on X. We shall say that λ is a* π-system *in case the support of λ_y is contained in $\pi^{-1}(y)$ for each $y \in Y$ and for each $f \in C_c(X)$, the function of y, $\lambda(f)(y) := \int f(x)\, d\lambda_y(x)$ lies in $C_c(Y)$. We shall say that a π-system is* full *in case the support of each λ_y is all of $\pi^{-1}(y)$.*

Thus every π-system determines a continuous linear map from $C_c(X)$ to $C_c(Y)$ — continuous in the inductive limit topologies — that is a bi-module map over $C_c(Y)$, where $C_c(Y)$ acts on $C_c(X)$ via the formula $f \cdot \varphi = f \circ \pi \varphi$. And conversely, it is not hard to see that every continuous $C_c(Y)$-bimodule map from $C_c(X)$ to $C_c(Y)$ that is continuous with respect to the inductive limit topologies (and maps non-negative functions to non-negative functions) is given by a π-system. For this reason, we shall often say that λ is a π-system *from* X *to* Y, or *from* $C_c(X)$ *to* $C_c(Y)$. As an example of a π-system, observe that a Haar system is one. Indeed here, $X = G$, $Y = G^{(0)}$, and $\pi = r$. (See the discussion after Definition 2.28.) Thus, in this case, a Haar system is an r-system.

Although we do not need the following fact for our immediate purposes, we state it here for future reference and for the purpose of showing that full π-systems exist. It is a corollary of a theorem of Blanchard, whose proof may be found in [**18**].

THEOREM 5.43. [**18**, Theorem 3.3] *If $\pi : X \to Y$ is a continuous open map from the separable, locally compact, Hausdorff space X onto the second countable, locally compact, Hausdorff space Y, then there is a full π-system of Radon measures on X.*

We note in passing that separability seems to be an essential assumption in this result.

LEMMA 5.44. [**174**, Lemma 1.1] *If $\pi : X \to Y$ is a continuous open surjection between locally compact Hausdorff spaces and if $\{\lambda_y\}_{y\in Y}$ is a full π-system on X, then the map $\lambda : C_c(X) \to C_c(Y)$ defined by the system is surjective.*

PROOF. Let $g \in C_c(Y)$ and let L be its support. Choose a compact set $K \subseteq X$ such that $\pi(K) = L$, select a function $f_1 \in C_c(X)$ that is strictly positive on K, and let $g_1 = \lambda(f_1)$. Then g_1 is strictly positive on L and $f := ((g/g_1) \circ \pi)f_1$ is mapped to g by λ. □

The next lemma is proved in essentially the same way, so the proof will be omitted.

LEMMA 5.45. [**136**, Lemma 2.14] *With the hypotheses of Lemma 5.44, suppose that U is an open subset of X, that g is a non-negative function in $C_c(X)$ that is supported in U, and that $\epsilon > 0$ is given. Then there is a non-negative function in $C_c(X)$ also supported in U such that*

$$|g(x) - f(x)\lambda(f) \circ \pi(x)| \leq \epsilon$$

for all $x \in X$.

Lemma 5.36 (part b) shows that if Z is a left principal G-space then a Haar system on G defines a π-system on Z, where $\pi : Z \to G\backslash Z$ is the quotient map. Moreover, the π-system is full. Specifically, we have the following lemma promised in Remark 5.37.

LEMMA 5.46. *If Z is a principal (left) G-space, if $\pi : Z \to G\backslash Z$ is the quotient map, and if $\tilde{\lambda}_{\pi(z)}$ is defined by the formula*

$$\int_Z f(x)\, d\tilde{\lambda}_{\pi(z)}(x) := \int_G f(\gamma^{-1}\cdot z)\, d\lambda^{r(z)}(\gamma) = \tilde{\lambda}(f)(\pi(z)),$$

$f \in C_c(Z)$, then $\{\tilde{\lambda}_{\pi(z)}\}_{\pi(z)\in G\backslash Z}$ is a full π-system. Consequently, by Lemma 5.44, $\tilde{\lambda} : C_c(Z) \to C_c(G\backslash Z)$ is surjective.

PROOF. The measure $\tilde{\lambda}_{\pi(z)}$ is the image of $\lambda^{r(z)}$ under the map from $G^{r(z)}$ to the orbit of z defined by: $\gamma \to \gamma^{-1} \cdot z$. The assumption that Z is a principal G-space guarantees that this map is a homeomorphism and the assumption that $\{\lambda^u\}_{u \in G^{(0)}}$ is a Haar system guarantees that $\tilde{\lambda}_{\pi(z)}$ is well defined. The fact that $\{\tilde{\lambda}_{\pi(z)}\}_{\pi(z) \in G \backslash Z}$ is a π-system follows from part b) of Lemma 5.36. The fact that it is full, i.e., that $\operatorname{supp} \tilde{\lambda}_{\pi(z)} = \pi^{-1}(\pi(z))$, follows from the facts that the orbit of z is homeomorphic to $G^{r(z)}$ and $\operatorname{supp} \lambda^{r(z)} = G^{r(z)}$. □

For the next lemma, recall from Lemma 2.36 that $G^{(0)}$ as a fundamental family of s-relatively compact neighborhoods.

LEMMA 5.47. [**171**, Proposition 2.1.9] *Let G be a locally compact groupoid of the kind we have been considering and suppose that G acts to the left on a locally compact Hausdorff space Z. Suppose that for each triple (K, U, ϵ) consisting of a compact subset $K \subseteq G^{(0)}$, an open s-relatively compact neighborhood, U, of $G^{(0)}$, and a positive number ϵ, there is an $e = e_{K,U,\epsilon} \in C_c(G)$, such that*

1. $e(\gamma) \geq 0$, *for all* γ;
2. $\operatorname{supp}(e) \subseteq U$; *and*
3. $\left|\int e(\gamma)\, d\lambda^u(\gamma) - 1\right| \leq \epsilon$, *for all* $u \in K$.

Then the family $\{e_{K,U,\epsilon}\}$, where the set $\{(K, U, \epsilon)\}$ is directed by increasing K, decreasing U, and decreasing ϵ, is a net that serves as a left approximate identity for $C_c(G)$ and for the action of $C_c(G)$ on $C_c(Z)$. If the $e_{K,U,\epsilon}$ can be chosen to be self-adjoint in $C_c(G)$, then $\{e_{K,U,\epsilon}\}$ is a two-sided approximate identity in $C_c(G)$.

PROOF. The last assertion clearly follows from the main part of the lemma, since taking adjoints is a homeomorphism of $C_c(G)$ in the inductive limit topology. We shall show that the family $\{e_{K,U,\epsilon}\}$ serves as an approximate identity for the left action of $C_c(G)$ on $C_c(Z)$, since substituting G for Z gives the proof for $C_c(G)$. Fix an initial s-relatively compact neighborhood U_0 of $G^{(0)}$. We shall take smaller and smaller U's contained in U_0. Let $\xi \in C_c(Z)$ $(\xi \neq 0)$ be given, let $K = \operatorname{supp} \xi$, and let $L = \overline{U_0 \cdot K}$. Then L is compact in Z, by Remark2.35. Given $\epsilon_0 > 0$, choose $\epsilon \leq \frac{1}{3} \min\{\epsilon_0 / \|\xi\|_\infty, \epsilon_0, 1\}$. Choose K such that $r(L) \subseteq K$, and choose U so that for $(\gamma, z) \in U \times L \subseteq G * Z$, $|\xi(\gamma^{-1}z) - \xi(z)| \leq \epsilon$. Then with $e = e_{K,U,\epsilon}$, both $\operatorname{supp} e * \xi$, and $\operatorname{supp} \xi$ are contained in L, and for $z \in L$,

$$
\begin{aligned}
(4.1) \qquad |e * \xi(z) - \xi(z)| &= \left| \int e(\gamma)\xi(\gamma^{-1}z)\, d\lambda^{r(z)}(\gamma) - \xi(z) \right| \\
&\leq \left| \int e(\gamma)(\xi(\gamma^{-1}z) - \xi(z))\, d\lambda^{r(z)}(\gamma) \right| + \left| (\int e(\gamma)\, d\lambda^{r(z)}(\gamma) - 1)\xi(z) \right| \\
&\leq \epsilon \int e(\gamma)\, d\lambda^{r(z)}(\gamma) + \epsilon \|\xi\|_\infty \\
&\leq \epsilon(1 + \epsilon) + \epsilon \|\xi\|_\infty \\
&\leq \frac{1}{3}(\epsilon_0(1 + \epsilon_0) + \epsilon_0) \\
&\leq \epsilon_0.
\end{aligned}
$$

□

PROOF OF PROPOSITION 5.39

We will consider only the left action of G on X. The proof for the right action of H is similar. Furthermore, we consider only the action of $C_c(G)$ on $C_c(X)$, since we could specialize X to G and use the fact that G always implements an equivalence between G and itself.

Observe first, that since X is a G-H-equivalence, $r : X \to G^{(0)}$ may be viewed as the quotient map, $X \to X/H$, and likewise, $s : X \to H^{(0)}$ may be viewed as the quotient map, $X \to G\backslash X$. Consequently, by Lemma 5.46, for $f \in C_c(X)$ and $u \in G^{(0)}$,

$$u \to \int_H f(x \cdot \eta)\, d\beta^{s(x)}(\eta),$$

where x is chosen so that $r(x) = u$, is a full r-system on X; and, likewise, for $v \in H^{(0)}$,

$$v \to \int_G f(\gamma^{-1} \cdot x)\, d\lambda^{r(x)}(\gamma),$$

where now x is chosen so that $s(x) = v$, is a full s-system on X. We will need both systems, even though we are focusing on the left action of G.

Suppose (K, U, ϵ) is given. By the properness of the G-action on X, we may find finitely many open, relatively compact sets V_i, $i = 1, \dots, n$, in X such that $\{r(V_i)\}_{i=1}^n$ covers K, and such that $\gamma \in U$, whenever $(\gamma \cdot x, x) \in V_i \times V_i$. We fix a partition of unity on $K \subseteq G^{(0)}$ that is subordinate to $\{r(V)\}_{i=1}^n$, and denote it by $\{b_i\}_{i=1}^n$. Thus, the b_i are assumed to be non-negative, to satisfy the relation $\mathrm{supp}(b_i) \subseteq r(V_i)$, and to satisfy the equation $\sum_{i=1}^n b_i(u) = 1$, for all $u \in K$. By Lemma 5.44 (more particularly Lemma 5.46), there are functions $\psi_i \in C_c(X)$ with the property that $\mathrm{supp}(\psi_i) \subseteq V_i$, such that

$$\int_H \psi_i(x \cdot \eta)\, d\beta^{s(x)}(\eta) = b_i(r(x))$$

for all $x \in X$. By Lemma 5.45, there are non-negative functions $\varphi_i \in C_c(X)$, with $\mathrm{supp}(\varphi_i) \subseteq V_i$, such that

$$\left| \psi_i(x) - \varphi_i(x) \int_G \varphi_i(\gamma^{-1} \cdot x)\, d\lambda^{r(x)}(\gamma) \right| \le \frac{\epsilon}{M},$$

where

$$M = \sup_x \sum_{i=1}^n \int 1_{V_i}(x \cdot \eta)\, d\beta^{s(x)}(\eta).$$

We define $e_{K,U,\epsilon} = e$ to be $\sum_{i=1}^n {}_A\langle \varphi_i, \varphi_i \rangle$, and claim that e satisfies the hypotheses of Lemma 5.47. Indeed, e is non-negative since the φ_i are non-negative. Also, note that e is self-adjoint, since ${}_A\langle \varphi, \varphi \rangle$ is self-adjoint for all $\varphi \in C_c(X)$. Since φ_i is supported in V_i, $e(\gamma) = 0$, for $\gamma \notin U$, by the selection of the V_i's. To verify the last condition of Lemma 5.47, suppose $u \in K$, and choose $x \in X$ such that $r(x) = u$.

Then

$$\begin{aligned}
&\left|\int_G e(\gamma)\,d\lambda^u(\gamma) - 1\right| \\
&= \left|\sum_{i=1}^n \int_G \int_H \varphi_i(x\eta)\varphi_i(\gamma^{-1}x\eta)\,d\beta^{s(x)}(\eta)\,d\lambda^{r(x)} - \sum_{i=1}^n b_i(r(x))\right| \\
&= \left|\sum_{i=1}^n \int_H \left\{\varphi_i(z\eta)\int_G \varphi_i(\gamma^{-1}x\eta)\,d\lambda^{r(x)}(\gamma) - \psi_i(x\eta)\right\}\,d\beta^{s(x)}(\eta)\right| \le \epsilon.
\end{aligned}$$

Finally, to obtain a sequence $\{e_n\}_{n=1}^\infty$ that is cofinal among the $\{e_{K,U,\epsilon}\}$, simply choose an expanding sequence of K's whose union is $G^{(0)}$, $\{K_n\}$, a decreasing sequence of U's whose intersection is $G^{(0)}$, $\{U_n\}$, a decreasing sequence of ϵ's converging to zero, $\{\epsilon_n\}$, and let $e_n = e_{K_n,U_n,\epsilon_n}$. □

5. Haar Systems on Imprimitivity Groupoids

We have remarked earlier that it may seem odd that no use was made of measures on X in the proof of Theorem 5.38. One of our goals here is to show that they are implicitly present. In fact, it will turn out that if X is a (right) principal G-space for a locally compact groupoid G with a Haar system, then the imprimitivity groupoid $X *_G X^{\mathrm{op}}$ has a Haar system if and only if there is a family of measures on X that constitute an equivariant s-system in the sense of the following definition.

Definition 5.48. *Suppose X and Y are locally compact, left G-spaces and that $\pi : X \to Y$ is a continuous, open,* equivariant map, *i.e., suppose that $\gamma \cdot \pi(x) = \pi(\gamma \cdot x)$ for all $(\gamma, x) \in G * X$. Then a π-system $\{\lambda_y\}_{y\in Y}$ on X is called* equivariant *in case for $(\gamma, y) \in G * Y$, $\gamma \cdot \lambda_y(f) = \lambda_{\gamma\cdot y}(f)$, for all $f \in C_c(X)$, i.e., in case $\int f(\gamma \cdot x)\,d\lambda_y(x) = \int f(x)\,d\lambda_{\gamma\cdot y}(x)$. Equivariant π-systems for right principal G-spaces are defined similarly.*

Of course, a Haar system is an example of an equivariant π-system. In this case, X is G, with G acting on the left by translation, and π is r. Recall that G acts on $G^{(0)}$ by the formula: $\gamma \cdot s(\gamma) = r(\gamma)$. Thus, in this case, a Haar system is an equivariant r-system. The following lemma was proved by Renault in [**174**].

Lemma 5.49. [**174**, Lemma 1.3] *Let X and Y be two locally compact (left) principal G-spaces and let $\pi : X \to Y$ be a continuous, open, equivariant surjection. Then π induces a continuous open surjection $\dot\pi : G\backslash X \to G\backslash Y$, and further:*

i) *Every equivariant π-system λ on X induces a $\dot\pi$-system $\dot\lambda$ on $G\backslash X$ according to the formula $\dot\lambda(f)(\dot y) = \int f(\dot x)\,d\dot\lambda_{\dot y}(\dot x) := \int f(\dot x)\,d\lambda_y(x)$, where $\dot x$ and $\dot y$ denote the images of x and y in $G\backslash X$ and $G\backslash Y$, respectively.*
ii) *Conversely, given a $\dot\pi$-system τ on $G\backslash X$, there is a unique equivariant π-system λ on X such that $\tau = \dot\lambda$.*

The proof of this lemma relies on another lemma of Renault's that will be useful.

Lemma 5.50. [**174**, Lemma 1.2] *Suppose that X, Y, and Z are locally compact Hausdorff spaces and suppose $\pi : X \to Z$ and $\tau : Y \to Z$ are continuous, open surjections. Let $X * Y = \{(x, y) \mid \pi(x) = \tau(y)\}$ and let $p_2 : X * Y \to Y$ be the projection onto the second component, $p_2(x, y) = y$. Then p_2 is a continuous* *open surjection. Suppose, further, that for each $z \in Z$, λ_z is a measure on X that is supported on $\pi^{-1}(z)$ and define λ_{2y} to be $\lambda_{\tau(y)} \times \epsilon_y$ (which is supported on $p_2^{-1}(y)$, $y \in Y$). Then λ is a continuous family of measures in the sense that $z \to \lambda(f)(z) := \int f(x)\, d\lambda_z(x)$ is continuous for all $f \in C_c(X)$ if and only if λ_2 is a continuous family of measures.*

Thus, except for an assertion about the supports of $\lambda(f)$ and $\lambda_2(f)$, λ is a π-system iff λ_2 is a p_2-system in the sense of Definition 5.42.

PROOF. It is easy to see that p_2 is continuous. The fact that it is open follows from Lemma 5.30. The fact that it is a surjection results from the surjectivity of π and τ. By the Stone-Weierstrass theorem we need only test λ_2 against functions of the form $F(x, y) = f(x)g(y)$, where $f \in C_c(X)$ and $g \in C_c(Y)$. But for such a function, $\lambda_2(F) = (\lambda(f) \circ \tau)g$. From this, it follows that λ is continuous iff λ_2 is continuous. □

PROOF OF LEMMA 5.49
Consider the diagram

$$\begin{array}{ccccc} & & \pi & & \\ & X & \longrightarrow & Y & \\ p & \downarrow & & \downarrow & q \\ & G\backslash X & \longrightarrow & G\backslash Y & \\ & & \dot{\pi} & & \end{array}$$

where p and q denote the quotient maps. Then, by definition, $\dot{\lambda}(f) \circ q = \lambda(f \circ p)$, $f \in C_c(G\backslash X)$.

Define $\phi : X \to (G\backslash X) * Y = \{(p(x), y) \mid \dot{\pi}(p(x)) = q(x)\}$ by $\phi(x) = (p(x), \pi(x))$. Then ϕ is an equivariant homeomorphism, where G acts on $(G\backslash X) * Y$ be translating in the second variable. Observe that $\phi(\pi^{-1}(y)) = \dot{\pi}^{-1}(q(y)) \times \{y\}$. Thus, p maps $\pi^{-1}(y)$ homeomorphically onto $\dot{\pi}(q(y))$. Since $\lambda_{\gamma y} = \gamma\lambda_y$, for all γ and y such that $s(\gamma) = r(y)$, the function $y \to p(\lambda_y)$ is constant on sets of the form $q^{-1}(z)$, $z \in G\backslash Y$. Set $\dot{\lambda}_z = p(\lambda_y)$, $z = q(y)$. Then $\phi(\lambda_y) = \dot{\lambda}_{q(y)} \times \epsilon_y$, and the continuity of $\dot{\lambda}$ follows from that of λ, by Lemma 5.50. Since, evidently, $\dot{\lambda}(f)$ has compact support, if f has compact support, we conclude that $\dot{\lambda}$ is a $\dot{\pi}$-system.

Conversely, given τ, note that since p maps $\pi^{-1}(y)$ homeomorphically onto $\dot{\pi}^{-1}(q(y))$, there is, for each $y \in Y$, a *unique* λ_y such that $\tau_{q(y)} = p(\lambda_y)$. If $s(\gamma) = r(y)$, then $q(\gamma y) = q(y)$ and for $x \in \pi^{-1}(y)$, $p(\gamma x) = p(x)$. Thus, $p(\gamma\lambda_y) = p(\lambda_y) = \tau_{q(y)} = \tau_{q(\gamma y)} = p(\lambda_{\gamma y})$, which shows that $\gamma\lambda_y = \lambda_{\gamma y}$. Thus, λ is equivariant. The fact that it is continuous follows from Lemma 5.50 again, and a straightforward check shows that $\lambda(f)$ has compact support for all $f \in C_c(X)$.

Suppose that X is a right principal G-space and let α be a full equivariant s-system on X. Define $\{\beta_x\}_{x \in X}$ by the formula $\beta_x := \epsilon_x \times \alpha_{s(x)}$ so that for $f \in C_c(X * X^{\mathrm{op}})$,

$$\int_{X * X^{\mathrm{op}}} f(u, v)\, d\beta_x(u, v) = \int_{X * X^{\mathrm{op}}} f(x, y)\, d\alpha_{s(x)}(y).$$

Then a moment's reflection reveals that $\{\beta_x\}_{x\in X}$ is a full p_1-system, where $p_1 : X * X^{\text{op}} \to X$ is projection onto the first variable. Moreover, since α is equivariant,

$$\begin{aligned}\int_{X*X^{\text{op}}} f(u,v)\,d\beta_{x\cdot\gamma}(u,v) &= \int_{X*X^{\text{op}}} f(x\cdot\gamma, y)\,d\alpha_{s(x\cdot\gamma)}(y)\\ &= \int_{X*X^{\text{op}}} f(x\cdot\gamma,\gamma^{-1}y)\,d\alpha_{s(x)}(y)\\ &= \int_{X*X^{\text{op}}} f(u\cdot\gamma,\gamma^{-1}v)\,d\beta_x(u,v),\end{aligned}$$

which shows that β is an equivariant p_1-system. By the first half of Lemma 5.49, β induces a $\dot p_1$-system $\dot\beta$ from $X *_G X^{\text{op}}$ to X/G. Now X/G is the unit space of the groupoid $X *_G X^{\text{op}}$ and it is not hard to see that if change notation and write $[x]$ for $\dot x$ in X/G, and if we set $\lambda^{[x]} = \dot\beta_{[x]}$, then λ is a Haar system on $X *_G X^{\text{op}}$. Indeed, the only thing that is at issue is the invariance of λ, but this is the result of the following equation:

$$\begin{aligned}\int_{X*_G X^{\text{op}}} f([x,y][u,v])\,d\lambda^{[y]}([u,v]) &= \int_{X*_G X^{\text{op}}} f([x,v])\,d\dot\beta_{[y]}\\ &= \int_{X*_G X^{\text{op}}} f([x,v])\,d\alpha_{s(y)}(v)\\ &= \int_{X*_G X^{\text{op}}} f([x,v])\,d\alpha_{s(x)}(v), \text{ since } s(x)=s(y),\\ &= \int_{X*_G X^{\text{op}}} f([x,v])\,d\lambda^{[x]}([x,v]).\end{aligned}$$

On the other hand, observe that a Haar system λ for $X *_G X^{\text{op}}$ is a $\dot p_1$-system. From the second half of Lemma 5.49, we conclude that there is a unique p_1-system β from $X * X^{\text{op}}$ to X such that $\dot\beta = \lambda$. From the left invariance of λ on $X *_G X^{\text{op}}$ and the uniqueness of β, it follows easily that β must be of the form $\beta_x = \epsilon_x \times \alpha_x$, with $\operatorname{supp}\alpha_x \subseteq s^{-1}(s(x))$. The equality $\operatorname{supp}\alpha_x = s^{-1}(s(x))$ follows from the equality $\operatorname{supp}\lambda^{[x]} = \{[x,y] \mid s(x) = r^{\text{op}}(y)\}$. Finally, the invariance of α and the fact that α_x depends only on $s(x)$ follows from the fact that $\lambda^{[x]}$ depends only on $[x]$ and not on x.

We have thus proved the following theorem, which is implicit in [**174**] and made explicit in [**104**].

THEOREM 5.51. [**104**, Proposition 5.2] *Let X be a principal right G-space. Given an equivariant s-system α on X, the formula*

$$\int_{X*_G X^{\text{op}}} f([u,v])\,d\lambda^{[x]}([u,v]) := \int_{X*X^{\text{op}}} f\circ\pi(u,v)\,d(\epsilon_x\times\alpha_{s(x)})(u,v), \tag{5.1}$$

where $f \in C_c(X *_G X^{\text{op}})$ and $\pi : X * X^{\text{op}} \to X *_G X^{\text{op}}$ is the quotient map, defines a Haar system λ on $X *_G X^{\text{op}}$; and conversely, if $X *_G X^{\text{op}}$ has a Haar system λ, then there is a unique equivariant s-system α on X such that equation (5.1) holds for all functions $f \in C_c(X *_G X^{\text{op}})$.

CHAPTER 6

Ideals, Orbits, and Amenability

In Chapter 1, we indicated that the ideal structure of a transformation group C^*-algebra is determined to a large extent by the orbit structure of the group action. The same is true for groupoid C^*-algebras. Our objective in this chapter is to report on the relationship between ideals and orbits. The relationship is the strongest when the groupoid is amenable and so we will survey some of the recent advances in the structure and analysis of amenable groupoids here also.

Throughout this chapter, our standing assumptions will be in force: all groupoids will be locally compact, Hausdorff, second countable, and have Haar systems.

1. Transitive Groupoids

Discuss representations induced from general closed subgroupoids in this section. Explain difficulties with Haar systems and how to compensate.

A transitive groupoid is one with only one orbit. So the structure of the associated C^*-algebra ought to be determined by the structure of the C^*-algebra of any isotropy group. Our objective in this section is to make this assertion precise.

So suppose G is a transitive, locally compact groupoid satisfying our standing hypotheses. Let $u \in G^{(0)}$ be a prescribed unit and let H be the isotropy group of u. Then, of course, H is a second countable, locally compact group. As was remarked in Examples 5.33 (see Example 3), the space G_u is a G-H equivalence, and so by Theorem 5.38, $C_c(G_u)$ may be completed to become a $C^*(G)$-$C^*(H)$ imprimitivity bimodule. This implies that the ideal structures of the two C^*-algebras $C^*(G)$ and $C^*(H)$ are "the same". (See [**156**] where it is shown how an ideal in $C^*(H)$ *induces* an ideal in $C^*(G)$ and vice-versa, and how this sets up a lattice isomorphism between the ideals in $C^*(G)$ and the ideals in $C^*(H)$.) While one might consider this to be the end of the story, it is possible to prove the following result that places the equivalence of $C^*(H)$ and $C^*(G)$ in a somewhat finer perspective and helps to show, too, that non-isomorphic groupoids can have isomorphic C^*-algebras. It is a generalization of a theorem of Phil Green [**83**] and of Corollary 3.34.

Fill in the theorem number

THEOREM 6.1. [**136**]*Suppose G is transitive and has a Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Let $u \in G^{(0)}$ and let $H = G|_{\{u\}}$ be the isotropy group of u. Then there is a positive measure μ on $G^{(0)}$ such that $C^*(G)$ is "naturally" isomorphic[1] to $C^*(H) \otimes K(L^2(G^{(0)}, \mu))$, where $K(L^2(G^{(0)}, \mu))$ denotes the C^*-algebra of compact operators on $L^2(G^{(0)}, \mu)$.*

PROOF. As we remarked above, G^u is a G-H-equivalence. Consequently, by Theorem 5.38, $C_c(G^u)$ may be completed to form an equivalence bimodule X_1 linking $C^*(G)$ with $C^*(H)$. We shall build another (left) Hilbert C^*-module X_2 over $C^*(H)$ that is isomorphic to X_1, say via a map $W : \mathsf{X}_1 \to \mathsf{X}_2$, and such that the

[1] The inverted commas are used here since the term 'natural' has a technical meaning implying that the isomorphism is independent of the choice of u and the cross section used in the proof of the theorem. The isomorphism depends on this data, but in a (non-technically) natural way.

imprimitivity algebra of X_2 is isomorphic to $C^*(H) \otimes K(L^2(G^{(0)}, \mu))$ for a suitable measure μ on $G^{(0)}$. Since $C^*(G)$ is the imprimitivity algebra of X_1, the isomorphism W will implement a C^*-isomorphism between $C^*(G)$ and $C^*(H) \otimes K(L^2(G^{(0)}, \mu))$. To form X_2, let $B_c(H) \otimes B_c(G^{(0)})$ denote the linear span of all functions of the form $\varphi \otimes \xi$, where $\varphi \otimes \xi(t, \omega) = \varphi(t)\xi(\omega)$ and φ and ξ are bounded, Borel, and compactly supported functions on H and $G^{(0)}$ respectively. Then $C_c(H)$ acts on $B_c(H) \otimes B_c(G^{(0)})$ via the formula $\psi \cdot (\varphi \otimes \xi) = (\psi * \varphi) \otimes \xi$, where $\psi * \varphi$ denotes the ordinary convolution of functions on H. For each Radon measure μ on $G^{(0)}$, we have a $C^*(H)$-valued inner product on $B_c(H) \otimes B_c(G^{(0)})$ defined by the formula

$$\langle \varphi_1 \otimes \xi_1, \varphi_2 \otimes \xi_2 \rangle(t) = \left(\int_H \varphi_1(ts)\overline{\varphi_2(s)}\, d\lambda_H(s) \right) \cdot \left(\int_{G^{(0)}} \xi_1(\omega)\overline{\xi_2(\omega)}\, d\mu(\omega) \right),$$

where λ_H is Haar measure on H. Observe that because H is a locally compact *group*, the values of this sesquilinear form lie in $C_c(H)$. It is then not difficult to prove that this inner product converts $B_c(H) \otimes B_c(G^{(0)})$ into a pre-Hilbert C^*-module over $C^*(H)$ whose completion, $\mathsf{X}(\mu)$, has the property that $\mathbb{K}(\mathsf{X}(\mu)) = C^*(H) \otimes K(L^2(G^{(0)}, \mu))$. So it remains to find a measure μ for which there is an isomorphism $W : \mathsf{X}_1 \to \mathsf{X}(\mu)$. By Lemma 4.12, there is a Borel cross section $\rho : G^{(0)} \to G^u$ to the map $s|G^u$ with the property that $\rho(K)$ has compact closure for each compact set $K \subseteq G^{(0)}$. Such a ρ is called a *regular* cross section to $s|G^u$. We shall prove that for each regular cross section ρ to $s|G^u$, there is a measure μ on $G^{(0)}$ and a $C^*(H)$-Hilbert module isometry from X_1 onto $\mathsf{X}(\mu)$. To this end,extend ρ to a map from G^u onto G^u by defining $\rho(x) = \rho(s(x))$ and define $\psi : G^u \to G$ by $\psi(x) = x\rho(x)^{-1}$. It is easy to check that ψ is well defined and that the range of ψ is contained in H. Now define $\varphi : G^u \to H \times G^{(0)}$ by $\varphi(x) = (\psi(x), s(x))$ and observe that φ is an equivariant Borel isomorphism for the action of H on $H \times G^{(0)}$ given by translation in the first variable. It follows that $\varphi(\lambda^u) = \lambda_H \times \mu$, for a certain measure μ on $G^{(0)}$. To see this, observe that since φ is equivariant and λ^u is invariant, $\varphi(\lambda^u)$ is an invariant measure on $H \times G^{(0)}$ for the action of H. Consequently, when $\varphi(\lambda^u)$ is disintegrated following the projection onto the first factor, using Theorem 3.6, we may write $\varphi(\lambda^u) = \int_{G^{(0)}} \nu_\omega\, d\tilde{\mu}(\omega)$, where $\tilde{\mu}$ is a measure on $G^{(0)}$ and each ν_ω is an H-invariant measure on H. Thus, each ν_ω is a multiple of Haar measure $C(\omega)\lambda_H$. The measurability of the map $\omega \to \nu_\omega$ implies the measurability of the function C and so, if μ is defined to be $C \cdot \tilde{\mu}$, then $\varphi(\lambda_H) = \lambda_H \times \mu$. (For more details, see [**187**].) It is a straightforward application of Fubini's theorem and the fact that the L^1-norm on $C_c(H)$ dominates the C^*-norm on $C_c(H)$ to show that the bounded Borel functions with compact support on G^u, $B_c(G^u)$, may be viewed as a dense subset of X_1. Likewise (more easily, actually), the compactly supported, bounded, Borel functions on $H \times G^{(0)}$ is dense in $\mathsf{X}(\mu)$. If W is defined by the formula $W\xi = \xi \circ \varphi^{-1}$, then W maps $B_c(G^u)$ bijectively onto $B_c(H \times G^{(0)})$ and routine computations show that W extends to be a $C^*(H)$-module isomorphism from X_1 onto $\mathsf{X}(\mu)$. □

2. Ideals in General

Show someplace in this section that the support of a quasi-invariant measure is invariant.

One of the main attractions of the theory of groupoids in operator algebra is the fact that an open invariant subset of the unit space of a locally compact groupoid determines an ideal in the C^*-algebra of the groupoid. The preceding section shows that not every ideal need come from an open invariant subset, but in the best of circumstances, which still are quite widely applicable, the ideal structure of a groupoid C^*-algebra can be "calculated" in terms of the structure of open invariant subsets of the unit space and the ideal structure of the C^*-algebras of isotropy groups. The following proposition was proved in [**171**] under restrictive hypotheses, which now are unnecessary by virtue of Renault's disintegration theorem, Theorem 3.32 It is the first step.

PROPOSITION 6.2. *Let G be a locally compact, second countable, Hausdorff groupoid, with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Let U be an open invariant subset of $G^{(0)}$, let $F = G^{(0)}\backslash U$, and let $I_c(U) = \{f \in C_c(G) \mid f(x) = 0,\ x \notin G|_U\}$. If $I(U)$ denotes the closure of $I_c(U)$ in $C^*(G)$, then $I(U)$ is an ideal in $C^*(G)$, isomorphic to $C^*(G|_U)$ and the quotient, $C^*(G)/I(U)$ is isomorphic to $C^*(G|_F)$. Thus, we have a short exact sequence*

$$0 \to C^*(G|_U) \to C^*(G) \to C^*(G|_F) \to 0. \tag{2.1}$$

PROOF. Define $j : C_c(G|_U) \to I_c(U)$ by the formula $j(f) = \tilde{f}$, where $\tilde{f}(x) = f(x)$, if $x \in G|_U$ and is zero otherwise. Also, let $p : C_c(G) \to C_c(G|_F)$ be the map defined by restricting functions in $C_c(G)$ to $G|_F$. Then j and p are continuous with respect to the inductive limit topology, and the following exact sequence is clear:

$$0 \to C_c(G|_U) \overset{j}{\to} C_c(G) \overset{p}{\to} C_c(G|_F) \to 0.$$

The proof is completed with an easy application of Theorem 3.32. □

REMARK 6.3. *If the restriction $G|_F$ is not measurewise amenable (see Definition 6.6 below) in Proposition 6.2, the short exact sequence 2.1 does not hold, if all the C^*-algebras are replaced by their reduced versions; i.e., one does* not *have, in general, a short exact sequence of the form*

$$0 \to C^*_{red}(G|_U) \to C^*_{red}(G) \to C^*_{red}(G|_F) \to 0.$$

An example, based on the tangent groupoid *of a manifold may be found in* [**175**, Remark 4.10].

Recall from Section 3.4 of [**54**] that if A is a C^*-algebra, if $\pi \in \mathrm{Rep}(A)$, and if S is a subset of $\mathrm{Rep}(A)$, the following two assertions are equivalent:

1. $\ker\pi \supseteq \cap\{\ker\sigma \mid \sigma \in S\}$.
2. Each vector state associated with π is a weak-$*$ limit of states that are sums of vector functionals associated to the representations σ in S.

If either assertion holds, we say that π is *weakly contained* in S. We also say that π_1 is weakly contained in π_2 if and only if π_1 is weakly contained in $\{\pi_2\}$. We say that π_1 and π_2 are *weakly equivalent* if and only if each is weakly contained in the other; this happens if and only if $\ker\pi_1 = \ker\pi_2$.

EXAMPLE 6.4. *If μ is a measure on the unit space of a locally compact groupoid G, then* $\mathrm{Ind}\,\mu$ *is weakly contained in* $\{\mathrm{Ind}\,\varepsilon_u\}_{u\in E}$, *where* $E = \mathrm{supp}\,\mu$ *because* $\mathrm{Ind}\,\mu$ *is unitarily equivalent to* $\int_{G^{(0)}} \mathrm{Ind}\,\varepsilon_u\, d\mu(u)$.

PROPOSITION 6.5. *With the notation of Proposition 6.2, let $I_{red}(U)$ denote the closure of $I_c(U)$ in $C^*_{red}(G)$.*

1. *If μ is a quasi-invariant measure on $G^{(0)}$ with support F, then* $\ker(\mathrm{Ind}\,\mu) = I_{red}(U)$.

2. *The map* $U \to I_{red}(U)$ *is an order preserving* injection *from the collection of open invariant subsets of* $G^{(0)}$ *into collection of ideals in* $C^*_{red}(G)$.

PROOF. 1. It suffices to assume that $F = G^{(0)}$ and to note that by Example 6.4, $\operatorname{Ind}\mu$ weakly contains any representation of the form $\operatorname{Ind}\mu'$ where $\operatorname{supp}\mu'$ is contained in $\operatorname{supp}\mu$.
2. This is clear from the first assertion.

□

The key ingredient for analyzing the reduced C^*-algebras of locally compact groupoids is the notion of amenability. We will have more to say about this in Section **??**. For now we use the following definition.

DEFINITION 6.6. *If* μ *is a quasi-invariant measure on* $G^{(0)}$, *we say that* μ *is* amenable *in case there is a net* $\{f_i\} \subseteq C_c(G)$ *satisfying the following two properties:*
1. *The functions* $u \to \int |f_i|^2 \, d\lambda^u$ *converge to the constant function* 1 *on* $G^{(0)}$ *in the weak-* topology of* $L^\infty(G^{(0)}, \mu)$.
2. *The functions* $f_i * f_i^*$ *converge to the constant function* 1 *on* G *in the weak-* topology of* $L^\infty(G, \nu)$, *where* ν *is the induced measure on* G, $\mu \circ \lambda$.

The groupoid G *is called* measurewise *amenable in case each quasi-invariant measure on* $G^{(0)}$ *is amenable.*

Here, the principal application of amenability is the following result proved by Renault [**171**, Proposition II.3.2].

THEOREM 6.7. *If* μ *is an amenable, quasi-invariant measure on* $G^{(0)}$ *and if* π *is the integrated form of a representation of* G *based on* μ, $(\mu, G^{(0)} * \mathcal{H}, L)$, *then* π *is weakly contained in* $\operatorname{Ind}\mu$.

PROOF. We merely present an outline, referring to [**171**, Proposition II.3.2] for details. Without loss of generality, we may assume that μ is a probability measure. Fix a unit vector ξ in $\int^{\oplus} \mathcal{H}(u)\, d\mu(u)$, and consider the vector state determined by ξ associated with π,

$$\phi(f) = (\pi(f)\xi, \xi) = \int f(y)(L(y)\xi \circ s(y), \xi \circ r(y))\, d\nu_0(y),$$

where, as usual, ν_0 denotes the symmetrized measure associated with $\nu = \mu \circ \lambda$: $\nu_0 = \Delta^{-1/2}\nu$, $\Delta = d\nu/d\nu^{-1}$. Form the functionals

$$\phi_i(f) := \int (f_i * f_i^*) f(y)(L(y)\xi \circ s(y), \xi \circ r(y))\, d\nu_0(y).$$

Then $\phi_i(f) \to \phi(f)$. However, $\phi_i(f) = (\operatorname{Ind} M(f)\xi_i, \xi_i)$ where $\xi_i(x) = \Delta^{1/2}(x)\overline{f_i(x)}L(x^{-1})\xi \circ r(x)$ is a section of the bundle $G^{(0)} * \mathcal{K}$, where $\mathcal{K}(u) = L^2(\lambda_u) \otimes \mathcal{H}(u)$, and $\operatorname{Ind} M$ is just the integrated form of $U \otimes I$ where U is the regular representation. This is a multiple of $\operatorname{Ind}\mu$, if $\dim \mathcal{H}(u)$ is constant, and is weakly equivalent to $\operatorname{Ind}\mu$ in any case. □

COROLLARY 6.8. *If the groupoid is measurewise amenable, then* $C^*(G) = C^*_{red}(G)$.

As in the case of locally compact groups, the converse of Corollary 6.8 is true. This was recently proved by Anantharaman–Delaroche and Renault in [**5**] and will be discussed in Section **??**. Our goal here is the following theorem, which is a corollary, Corollary 4.9, of Renault's generalization [**175**] of the Gootman-Rosenberg solution [**79**] to the so-called Effros-Hahn conjecture [**63**].

THEOREM 6.9. [**175**, Corollary 4.9]*Suppose that the groupoid G is principal and measurewise amenable. Then the map $U \to I(U)$, from the collection of open invariant subsets of $G^{(0)}$ to the ideals in $C^*(G)$, is a bijection.*

PROOF. Again, we merely sketch the ideas of the proof referring to [**175**] for details. By Corollary 6.8 and Proposition 6.5, we know the map is injective. We need to show the map is surjective. So, given an ideal I, choose a representation π of $C^*(G)$ so that $I = \ker \pi$ and write π as the integrated form of a representation of G, $(\mu, G^{(0)} * \mathcal{H}, L)$. We saw in Theorem 6.7 that π is weakly contained in $\operatorname{Ind}\mu$. If we knew that π is weakly equivalent to $\operatorname{Ind}\mu$, the proof would be complete, by Proposition 6.5 and Corollary 6.8. Now $\operatorname{Ind}\mu$ is weakly equivalent to the representation called $\operatorname{Ind} M$ in Theorem 6.7. Here, we need to be a bit more explicit about the form of $\operatorname{Ind} M$. By hypothesis, $(\pi(f)\xi, \xi) = \int f(x)(L(x)\xi \circ s(x), \xi \circ r(x))\, d\nu_0(x)$, $\xi \in H := \int^{\oplus} \mathcal{H}(u)\, d\mu(u)$. Define the representation M on this Hilbert space by the formula $M(f)\xi(u) = f(u)\xi(u)$. Let $P : C_c(G) \to C_c(G^{(0)})$ be defined by the formula $P(f) = f|G^{(0)}$. (Then P is what is known as a generalized conditional expectation.) Define the sesquilinear form on $C_c(G) \otimes H$ by the formula

$$(f_1 \otimes \xi_1, f_2 \otimes \xi_2) = (\xi_1, M \circ P(f_1^* * f_2)\xi_2).$$

This form is semi-definite and gives, in the usual way, a Hilbert space K. The representation $\operatorname{Ind} M$ acts on K through the formula $\operatorname{Ind} M(f)(g \otimes \xi) = f * g \otimes \xi$. We remarked in the proof of Theorem 6.7 that $\operatorname{Ind} M$ is weakly equivalent to $\operatorname{Ind}\mu$. So we need only show that $\operatorname{Ind} M$ is weakly contained in π. Write $\hat{\pi}$ for $\operatorname{Ind} M$. Then we have

$$(f_1 \otimes \xi_1, \hat{\pi}(f)(f_2 \otimes \xi_2)) = (\xi_1, M \circ P(f_1^* * f * f_2)\xi_2).$$

In Lemma 3.2 of [**175**], Renault proves that there is a uniformly bounded net $\{Q_\alpha\}$ of maps on $C^*(G)$ of the form

$$Q_\alpha(f) = \sum_{i=1}^{n_\alpha} e_i^\alpha f e_i^\alpha,$$

where each $e_i^\alpha \in C_0(G^{(0)})$ and $e_i^\alpha f e_i^\alpha(x) = e_i^\alpha(r(x))f(x)e_i^\alpha(s(x))$, $x \in G$, such that $\pi \circ Q_\alpha(f) \to M \circ P(f)$ weakly for each $f \in C_c(G)$. This shows that π weakly contains $\operatorname{Ind} M$.

REMARK 6.10. *In the r-discrete case, the proof of Theorem 6.9 is substantially easier. See* [**171**, Proposition II.4.6].

□

COROLLARY 6.11. *If G is a measurewise principal groupoid, then $C^*(G)$ is simple if and only if G is minimal, i.e., if and only if there are no proper open invariant subsets of $G^{(0)}$.*

REMARK 6.12. *Both Theorem 6.9 and Corollary 6.11 depend heavily on the hypothesis that the groupoid G is Hausdorff. In an appendix to* [**175**], *G. Skandalis exhibits a minimal foliation with a non-Hausdorff holonomy groupoid G such that $C^*(G)$ is* not *simple.*

Say something about the effect of isotropy here and discuss the Gootman-Rosenberg solution to the Effros-Hahn conjecture.

3. Measure-Theoretically Smooth Groupoids

Give a list of all the equivalent conditions?

Recall that a C^*-algebra A is type I in case for each representation π of A on a Hilbert space H, the weakly closed algebra generated by $\pi(A)$ is a type I von Neumann algebra. This happens if and only if for each *irreducible* representation π of A on H, $\pi(A)$ contains the full ideal of compact operators on H. Of particular importance for the material we are discussing is the fact that A is type I precisely when an irreducible representation is uniquely determined by its kernel. That is, A is type I if and only if for any two irreducible representations π_i on Hilbert spaces H_i, $i = 1, 2$, the equation $\ker \pi_1 = \ker \pi_2$ if and only if π_1 is unitarily equivalent to π_2. In this event, Prim(A) with the Jacobson topology is homeomorphic to the spectrum $\hat{A}$ with the topology inherited from the pointwise-weak operator topology on Rep(A).

Add in the discussion about isotropy

In the case when A is the C^*-algebra of a locally compact groupoid, G, then, Theorem 6.16 implies that when $C^*(G)$ is type I, then $\widehat{C^*(G)}$ as a set is completely determined by the spectra of the isotropy groups of G. It is, therefore, important to determine when $C^*(G)$ is type I. The following theorem, due to Ramsay [**163**], generalizes earlier results of Glimm [**77**] and Effros [**60**] that were proved in the context of transformation groups.

THEOREM 6.13. [**163**]*Let G be a Polish groupoid and let $R = (r, s)(G)$ be the associated equivalence relation in $G^{(0)} \times G^{(0)}$. Then the following conditions are equivalent:*

1. *For each unit $u \in G^{(0)}$, the restriction of r to $G_u (= s^{-1}(u))$ induces a homeomorphism between $G_u/(G|_{\{u\}})$ and the orbit of u.*
2. *Each orbit is a G_δ subset of $G^{(0)}$.*
3. *The quotient space $G^{(0)}/G$ of all orbits with the quotient topology is T_0.*

Further, if R is an F_σ subset of $G^{(0)} \times G^{(0)}$, in particular, if G is a second countable, locally compact, Hausdorff groupoid, then these conditions are equivalent to each of the following:

4. *Each orbit is locally closed.*
5. *The quotient Borel structure on $G^{(0)}/G$ is countably separated.*
6. *Every (not–necessarily–quasi-invariant) ergodic measure on $G^{(0)}$ is concentrated on an orbit; i.e., if a measure μ has the property that each invariant Borel set is either null or co-null, then* supp μ *is contained in an orbit.*
7. *$G^{(0)}/G$ is a standard Borel space.*
8. *There is a Borel cross section to the quotient map $\pi : G^{(0)} \to G^{(0)}/G$.*
9. *The equivalence relation $R \subseteq G^{(0)} \times G^{(0)}$ is a G_δ subset.*

REMARK 6.14. *In Ramsay's theorem, there are five other conditions that are each equivalent to the ones presented.*

DEFINITION 6.15. *A Polish groupoid that satisfies any, and hence all, of the equivalent conditions in Theorem 6.13 is called* measure theoretically smooth. *The groupoid is also said to have a* smooth orbit space.

THEOREM 6.16. *Let G be a locally compact, second countable groupoid with Haar system. Then $C^*(G)$ is a type I C^*-algebra if and only if G has a smooth orbit space and for each $u \in G^{(0)}$, the isotropy group $G|_{\{u\}}$ is type I.*

CHAPTER 7

Bundles and Groupoid Crossed Products

This chapter will deal with C^*-bundles, groupoids acting on C^*-bundles, and the resulting crossed products.

CHAPTER 8

Coordinatization and Examples

This chapter will deal with coordinatization theorems and examples of a diverse number of groupoids.

CHAPTER 9

Triangular Operator Algebras

This chapter will deal with partial orders in groupoids and triangular operator algebras based on them.

CHAPTER 10

Representations and Dilations

This chapter will deal with the representation theory of triangular operator algebras.

Bibliography

[1] S. Adams. An equivalence relation that is not freely generated. *Proc. Amer. Math. Soc.*, 102:565–566, 1988.

[2] W. Ambrose. Representation of ergodic flows. *Ann. Math.*, 42:723–739, 1941.

[3] C. Anantharaman-Delaroche. Systemes dynamiques noncommutatifs et moyennabilité. *Math. Ann.*, 279:297–315, 1987.

[4] C. Anantharaman-Delaroche. Purely infinite C^*-algebras arising from dynamical systems. *Bull. Soc. math. France*, 125:199–225, 1997.

[5] C. Anantharaman-Delaroche and J. Renault. Amenable groupoids. Preprint, Université d'Orléans, 1997.

[6] M. A. Aof and R. Brown. The holonomy groupoid of a locally topological groupoid. *Topology Appl.*, 47:97–113, 1992.

[7] Wm. Arveson. Subalgebras of C^*-algebras. *Acta Math.*, 123:141–224, 1969.

[8] Wm. Arveson. Subalgebras of C^*-algebras. II. *Acta Math.*, 128:271–308, 1972.

[9] Wm. Arveson. Operator algebras and invariant subspaces. *Ann. Math.*, 100:433–532, 1974.

[10] Wm. Arveson. *An Invitation to C^*-Algebra*, volume 39 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1976.

[11] V. Arzumanian and J. Renault. Examples of pseudogroups and their C^*-algebras. In S. Doplicher, R. Longo, J. E. Roberts, and L. Zsido, editors, *Operator Algebras and Quantum Field Theory*, pages 93–104. International Press, 1997.

[12] V. Arzumanian and A. Vershik. Star-algebras associated with endomorphisms. In *Operator Algebras and Group Representations*, pages 17–27, Boston, 1984. Pitman.

[13] L. Auslander and C. Moore. Unitary representations of solvable Lie groups. *Mem. Amer. Math. Soc.*, 62, 1966.

[14] L. Baggett and A. Ramsay. A functional analytic proof of a selection lemma. *Canad. J. Math.*, 32:441–448, 1980.

[15] W. Beer. On Morita equivalence of nuclear C^*-algebras. *J. Pure Appl. Algebra*, 26:249–267, 1982.

[16] J. Bellisard. *K-Theory for C^*-Algebras in Solid State Physics*, volume 257 of *Lecture Notes in Physics*, pages 99–256. Springer-Verlag, Berlin, New York, Heidelberg, 1986.

[17] B. Blackadar. *K-Theory for Operator Algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, 2nd edition, 1998.

[18] E. Blanchard. Déformations de C^*-algèbres de Hopf. *Bull. Soc. Math. France*, 124:141–215, 1996.

[19] D. Blecher. A completely bounded characterization of operator algebras. *Math. Ann.*, 303:227–239, 1995.

[20] D. Blecher. A generalization of Hilbert modules. *J. Functional Anal.*, 136:365–421, 1996.

[21] D. Blecher, P. Muhly, and V. Paulsen. Categories of operator modules (Morita equivalence and projective modules). *Mem. Amer. Math. Soc.*, 1999.

[22] D. Blecher, Z-J. Ruan, and A. Sinclair. A characterization of operator algebras. *J. Functional Anal.*, 89:188–201, 1990.

[23] J.-B. Bost and A. Connes. Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Math. (New Series)*, 1:411–457, 1995.

[24] N. Bourbaki. *Intégration*, volume 6, Chap. 6 of *Élements de Mathématique*. Hermann, Paris, 1959.

[25] N. Bourbaki. *General Topology, Part 1*. Addison-Wesley, Reading, Mass., 1966.

[26] W. Brandt. über eine Verallgemeinerung des Gruppenbegriffes. *Math. Ann.*, 96:360–366, 1927.

[27] B. Brenken. The local product structure of expansive automorphisms of solenoids and their associated C^*-algebras. *Canadian J. Math.*, 48:692–709, 1996.

[28] L. Brown, P. Green, and M. Rieffel. Stable isomorphisms and strong Morita equivalence of C^*-algebras. *Pacific J. Math.*, 71:349–363, 1977.

[29] R. Brown. From groups to groupoids: a brief survey. *Bull. London Math. Soc.*, 19:113–134, 1987.

[30] R. Brown. *Topology: A Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid.* Mathematics and its Applications. Ellis Horwood Limited, Chichester, 1988.

[31] J-L. Brylinski. *Loop Spaces, Characteristic Classes, and Geometric Quantization*, volume 107 of *Progress in Mathematics.* Birkhäuser, Boston, 1992.

[32] J-L. Brylinski and V. Nistor. Cyclic cohomology of Étale groupoids. *K-theory*, 8:341–365, 1994.

[33] C. Camacho and A. Neto. *Geometric Theory of Foliations.* Birkhäuser, Boston, 1985.

[34] F. Combes. Crossed products and Morita equivalence. *Proc. London Math. Soc. (3)*, 49:289–306, 1984.

[35] A. Connes. *Sur la théorie non commutative de l'intégration*, volume 725 of *Lecture Notes in Mathematics*, pages 19–143. Springer-Verlag, Berlin, 1979.

[36] A. Connes. *A survey of foliations and operator algebras*, volume 38 of *Proc. Sympos. Pure Math.*, pages 521–628. Amer. Math. Soc., Providence, 1982.

[37] A. Connes. *Noncommutative Geometry.* Academic Press, Inc., San Diego, 1994.

[38] A. Connes and M. Takesaki. The flow of weights on factors of type III. *Tôhoku Math. J.*, 29:473–575, 1977. Eratta. ibid. **30** (1978), 653–655.

[39] E. Contensou. *C^*-Algèbres Des Quasi-Représentations D'un Groupe Discret et Quasi-cristaux.* PhD thesis, Université Paul Sabatier Toulouse, 1996.

[40] J. Cuntz. Simple C^*-algebras generated by isometries. *Commun. Math. Phys.*, 57:173–185, 1977.

[41] J. Cuntz. On the homotopy groups of the space of endomorphisms of a C^*-algebra (with applications to topological Markov chains). In *Operator Algebras and Group Representations*, pages 124–137, Boston, 1984. Pitman.

[42] J. Cuntz and W. Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.*, 56:251–268, 1980.

[43] R. Curto, P. Muhly, and D. Williams. Cross products of strongly Morita equivalent C^*-algebras. *Proc. Amer. Math. Soc.*, 90:528–530, 1984.

[44] M. Dakin and A. Seda. g-spaces and topological groupoids. *Glasnik Mat. Ser. III*, 12(32):191–198, 1977.

[45] K. Davidson. *Nest Algebras*, volume 191 of *Pitman Research Notes in Mathematics.* Longman Scientific and Technical, Essex, 1988.

[46] K. Davidson. When locally contractive representations are completely contractive. *J. Functional Anal.*, 128:186–225, 1995.

[47] K. Davidson, V. Paulsen, and S. Power. Tree algebras, semidiscreteness, and dilation theory. *Proc. London Math. Soc. (Series 3)*, 68:178–202, 1994.

[48] V. Deaconu. Embeddings of toral algebras and the corresponding Cuntz-Krieger algebras. Preprint, University of Nevada Reno, 1996.

[49] V. Deaconu. Generalized solinoids and c^*-algebras. *Pacific J. Math.*

[50] V. Deaconu. Groupoids associated with endomorphisms. *Trans. Amer. Math. Soc.*, 347:1779–1786, 1995.

[51] V. Deaconu. A path model for circle algebras. *J. Operator Theory*, 34:57–89, 1995.

[52] V. Deaconu. Generalized Cuntz-Krieger algebras. *Proc. Amer. Math. Soc.*, 124:3427–3435, 1996.

[53] J. Dieudonné. Sur le théorème de Lebesgue-Nikodym. III. *Ann. Univ. Grenoble (N.S.)*, 23:25–53, 1948.

[54] J. Dixmier. *C^*-Algebras.* Elsevier North-Holland, Inc., New York, 1977.

[55] J. Dixmier. *von Neumann Algebras.* Elsevier North Holland, Inc., New York, 1981.

[56] P. Doubillet, G. Rota, and R. Stanley. On the foundations of combinatorial theory (VI): The idea of a generating function. In *Proc. Sixth Berkeley Symposium in Math., Stat., and Prob., Vol. II*, pages 267–318, 1971.

[57] R. Douglas and V. Paulsen. *Hilbert Modules over Function Algebras*, volume 217 of *Pitman Research Notes in Mathematics Series*. Longman Scientific and Technical, Essex, 1989.

[58] H. Dye. On groups of measure preserving transformations. I. *Amer. J. Math.*, 81:119–159, 1959.

[59] H. Dye. On groups of measure preserving transformations. II. *Amer. J. Math.*, 85:551–576, 1963.

[60] E. Effros. Transformation groups and C^*-algebras. *Ann. of Math.(2)*, 81:38–55, 1965.

[61] E. Effros. Global structure in von Neumann algebras. *Trans. Amer. Math. Soc.*, 121:434–454, 1966.

[62] E. Effros. Advances in quantized functional analysis. In *Proc. ICM Berkeley*, 1986.

[63] E. Effros and F. Hahn. Locally compact transformation groups and C^*-algebras. *Memoirs Amer. Math. Soc.*, 75, 1967.

[64] T. Fack. Quelques remarques sur le spectre des C^*-algèbres de feuillages. *Bull. Soc. Math. Belg. Sér. B*, 36:113–129, 1984.

[65] H. Federer and A. Morse. Some properties of measurable functions. *Bull. Amer. Math. Soc.*, 49:270–277, 1943.

[66] J. Feldman, P. Hahn, and C. Moore. Orbit structure and countable sections for actions of continuous groups. *J. Functional Anal.*, 28:186–230, 1978.

[67] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Amer. Math. Soc.*, 234:289–324, 1977.

[68] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. II. *Trans. Amer. Math. Soc.*, 234:325–359, 1977.

[69] J. Feldman and A. Ramsay. Countable sections for free actions of groups. *Adv. in Math.*, 55:224–227, 1985.

[70] J. Fell and R. Doran. *Representation of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles. Vol. 1*, volume 125 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1988.

[71] J. Fell and R. Doran. *Representation of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles. Vol. 2*, volume 126 of *Pure and Applied Mathematics*. Academic Press, Inc., San Diego, 1988.

[72] C. Foiaş and A. Frazho. *The Commutant Lifting Approach to Interpolation Problems*, volume 44 of *Operator Theory: Advances and Applications*. Birkhäuser, Basel-Boston-Berlin, 1990.

[73] P. Forrest. Virtual subgroups of $\mathbb{R}^n$ and $\mathbb{Z}^n$. *Adv. in Math.*, 3:187–207, 1974.

[74] P. Gabriel. Unzerlegbare darstellungen I. *Manuscr. Math.*, 6:71–103, 1972.

[75] P. Gabriel. *Representations of Finite-Dimensional Algebras*, volume 73 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, New York, 1992.

[76] P.-Y. Le Gall. *Théorie de Kasparov équivariante et groupoïdes*. PhD thesis, Université de Paris VII, June 1994.

[77] J. Glimm. Locally compact transformation groups. *Trans. Amer. Math. Soc.*, 101:124–138, 1961.

[78] C. Goffman. *Real Functions*. Rinehart & Co., Inc, New York, N. Y., 1953.

[79] E. Gootman and J. Rosenberg. The structure of crossed product C^*-algebras: a proof of the generalized Effros-Hahn conjecture. *Invent. Math.*, 52:283–298, 1979.

[80] P. Green. C^*-algebras of transformation groups with smooth orbit space. *Pacific J. Math.*, 72:71–97, 1977.

[81] P. Green. The brauer group of a commutative C^*-algebra. Unpublished seminar notes from the University of Pennsylvania, 1978.

[82] P. Green. The local structure of twisted covariance algebras. *Acta mathematica*, 140:191–250, 1978.

[83] P. Green. The structure of imprimitivity algebras. *J. Funct. Anal.*, 36:88–104, 1980.

[84] F. Greenleaf. *Invariant Means on Topological Groups*, volume 16 of *Van Nostrand Mathematical Studies*. Van Nostrand-Reinhold Co., New York, 1969.

[85] A. Haefliger. Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. *Comm. Math. Helv.*, 32:248–329, 1958.

[86] A. Haefliger. *Groupoïde d'holonomie et classifiant*, volume 116 of *Astérisque*, pages 70–97. Société Mathématique de France, 1984.

[87] P. Hahn. Haar measure for measure groupoids. *Trans. Amer. Math. Soc.*, 242:1–33, 1978.

[88] P. Hahn. The regular representation of measure groupoids. *Trans. Amer. Math. Soc.*, 242:35–72, 1978.

[89] P. Hahn. Reconstruction of a factor from measures on Takesaki's unitary equivalence relation. *J. Functional Anal.*, 31:263–271, 1979.

[90] P. Halmos. The decomposition of measures. *Duke Math. J.*, 8:386–392, 1941.

[91] M. Hamana. Injective envelopes of operator systems. *Publ. Res. Inst. Math. Sci., Kyoto Univ.*, 15:773–785, 1979.

[92] P. Higgins. *Notes on Categories and Groupoids*, volume 32 of *Mathematical Studies*. Van Nostrand Reinhold Co., London, 1971.

[93] M. Hilsum and G. Skandalis. Stabilité des C^*-algèbres de feuilletages. *Ann. Inst. Fourier (Grenoble)*, 33:201–208, 1983.

[94] M. Hilsum and G. Skandalis. Morphisms K-orient'es d'éspaces de feuilles et fonctorialité en théorie de kasparov (d'après un conjecture d'A. Connes). *Ann. Scient. 'Ecole. Norm. Sup.*, 20:325–390, 1987.

[95] J.Renault. The Fourier algebra of a measured groupoid and its multipliers. *J. Functional Anal.*, 145:455–490, 1997.

[96] I. Kaplansky. Modules over operator algebras. *Trans. Amer. Math. Soc.*, 75:839–858, 1953.

[97] G. Kasparov. Hilbert C^*-modules: Theorems of Stinespring and Voiculescu. *J. Operator Theory*, 4:133–150, 1980.

[98] D. Kastler. Connes' noncommutative integration theory. *Commun. Math. Phys.*, 85:99–120, 1982.

[99] J. Kellendonk. Non-commutative geometry of tilings and gap labeling. *Reb. Math. Phys.*, 7:1133–1180, 1995.

[100] A. Kumjian. *Diagonals in algebras of continuous trace*, volume 1132 of *Lecture Notes in Mathematics*, pages 297–311. Springer-Verlag, New York, 1983.

[101] A. Kumjian. On localizations and simple C^*-algebras. *Pacific J. Math.*, 112:141–192, 1984.

[102] A. Kumjian. On C^*-diagonals. *Canad. J. Math.*, 38:969–1008, 1986.

[103] A. Kumjian. On equivariant sheaf cohomology and elementary C^*-bundles. *J. Operator Theory*, 20:207–240, 1988.

[104] A. Kumjian, P. Muhly, J. Renault, and D. Williams. The Brauer group of a locally compact groupoid. *Amer. J. Math.*, 120:901–954, 1998.

[105] A. Kumjian, D. Pask, and I. Raeburn. Cuntz-Krieger algebras of directed graphs. *Pacific J. Math.*, 184:161–174, 1998.

[106] A. Kumjian, D. Pask, I. Raeburn, and J. Renault. Graphs, groupoids, and Cuntz-Krieger algebras. *J. Functional Anal.*, 144:505–541, 1997.

[107] A. Kumjian, I. Raeburn, and D. Williams. The equivariant Brauer groups of commuting free and proper actions are isomorphic. *Proc. Amer. Math. Soc.*, 124:809–817, 1996.

[108] K. Kuratowski. *Topology*. Academic Press, New York and London, 1966. English translation, revised and augmented, of the fourth French edition.

[109] E. C. Lance. *Hilbert C^*-modules: A toolkit for operator algebraists*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[110] N. P. Landsman. *Mathematical Topics Between Classical and Quantum Mechanics*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998.

[111] N. P. Landsmann. Strict deformation quantization of a particle in external gravitational and Yang-Mills fields. *J. Geometry and Physics*, 12:93–132, 1993.

[112] M. Macho-Stadler. *Isomorphise de Thom pour les feuilletages presque sans holonomie*. PhD thesis, Université Claude Bernard Lyon I, 1996.

[113] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[114] G. Mackey. Imprimitivity for representations of locally compact groups. I. *Proc. Nat. Acad. Sci. U.S.A.*, 35:537–545, 1949.

[115] G. Mackey. Induced representations of groups. I. *Ann. Math.*, 55:101–139, 1952.

[116] G. Mackey. Induced representations of locally compact groups. II, the Frobenius reciprocity theorem. *Ann. Math.*, 58:193–220, 1953.

[117] G. Mackey. Borel structures in groups and their duals. *Trans. Amer. Math. Soc.*, 85:134–165, 1957.

[118] G. Mackey. Unitary representations of group extensions I. *Acta Math.*, 99:265–311, 1958.

[119] G. Mackey. Point realizations of transformation groups. *Illinois J. Math.*, 6:327–335, 1962.

[120] G. Mackey. Ergodic theory, group theory, and differential geometry. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1184–1191, 1963.

[121] G. Mackey. Infinite dimensional group representations. *Bull. Amer. Math. Soc.*, 69:628–686, 1963.

[122] G. Mackey. *Mathematical Foundations of Quantum Mechanics*. The Mathematical Physics Monograph Series. W. A. Benjamin, Inc., New York, 1963.

[123] G. Mackey. Ergodic theory and virtual groups. *Math. Ann.*, 166:187–207, 1966.

[124] G. Mackey. Ergodic theory and its significance for statistical mechanics and probability theory. *Adv. in Math.*, 12:178–268, 1974.

[125] G. Mackey. *The Theory of Unitary Group Representations*. Chicago Lectures in Mathematics Series. The University of Chicago Press, Chicago, 1976. An expanded version of notes on lectures given at the University of Chicago in 1955, and circulated in mimeographed form under the title: "The Theory of Group Representations".

[126] T. Masuda. Groupoid dynamical systems and crossed product, I-The case of W^*-systems. *Publ. Res. Inst. Math. Sci., Kyoto Univ.*, 20:929–957, 1984.

[127] T. Masuda. Groupoid dynamical systems and crossed products, II: the case of C^*-systems. *Publ. Res. Inst. Math. Sci., Kyoto Univ.*, 20:959–970, 1984.

[128] R. Mercer. The full group of a countable measurable equivalence relation. *Proc. Amer. Math. Soc.*, 117:323–333, 1993.

[129] I. Moerdick and D. Pronk. Orbifolds, sheaves, and groupoids. *K-theory*, 12:3–21, 1997.

[130] C. Moore. Group extensions and cohomology for locally compact groups. III. *Trans. Amer. Math. Soc.*, 221:1–33, 1976.

[131] C. Moore. Group extensions and cohomology for locally compact groups. IV. *Trans. Amer. Math. Soc.*, 221:34–58, 1976.

[132] C. Moore and C. Schochet. *Global Analysis on Foliated Spaces*, volume 9 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1988.

[133] K. Morita. Duality of modules and its applications to the theory of rings with minimum condition. *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A*, 6:85–142, 1958.

[134] P. Muhly, C. Qiu, and B. Solel. Coordinates, nuclearity, and spectral subspaces in operator algebras. *J. Operator Theory*, 26:313–332, 1991.

[135] P. Muhly and J. Renault. C^*-algebras of multivariable Wiener-Hopf operators. *Trans. Amer. Math. Soc.*, 274:1–44, 1982.

[136] P. Muhly, J. Renault, and D. Williams. Equivalence and isomorphism for groupoid C^*-algebras. *J. Operator Theory*, 17:3–22, 1987.

[137] P. Muhly, J. Renault, and D. Williams. Continuous-trace groupoid C^*-algebras. III. *Trans. Amer. Math. Soc.*, 348:3621–3641, 1996.

[138] P. Muhly and D. Williams. Continuous trace groupoid C^*-algebras. *Math. Scand.*, 66:231–241, 1990.

[139] P. Muhly and D. Williams. Continuous trace groupoid C^*-algebras. II. *Math. Scand.*, 70:127–145, 1992.

[140] P. Muhly and D. Williams. Groupoid cohomology and the Dixmier-Douady class. *Proc. London Math. Soc. (3)*, 71:109–134, 1995.

[141] J. von Neumann. Zur Operatorenmethode in der klassischen Mechanik. *Ann. Math.*, 33:789–791, 1932.

[142] K. Parthasarathy. *Probability Measures on Metric Spaces*. Academic Press, New York, 1967.

[143] W. Paschke. Inner product modules over B^*-algebras. *Trans. Amer. Math. Soc.*, 182:443–468, 1973.

[144] D. Pask. Cuntz-Krieger algebras associated to directed graphs. In S. Doplicher, R. Longo, J.E. Roberts, and L. Zsido, editors, *Operator Algebras and Quantum Field Theory*, pages 85–92. International Press, 1997.

[145] A. Paterson. *Groupoids, Inverse Semigroups, and their Operator Algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser, Boston, 1999.

[146] A. Patterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, 1988.

[147] G. Pedersen. *C^*-algebras and their Automorphism Groups*. Academic Press, London, 1979.

[148] J.-P. Pier. *Amenable Locally Compact Groups*. Wiley, New York, 1984.

[149] I. Putnam. C^*-algebras from Smale spaces. *Can. J. Math.*, 48:175–195, 1996.

[150] I. Putnam. Hyperbolic systems and generalized Cuntz-Krieger algebras. Lecture notes from the Summer School in Operator Algebras, Odense, Denmark, 1996.

[151] I. Raeburn. On the Picard group of a continuous-trace C^*-algebra. *Trans. Amer. Math. Soc.*, 263:183–205, 1981.

[152] I. Raeburn and J. Rosenberg. Crossed products of continuous-trace C^*-algebras. *Trans. Amer. Math. Soc.*, 305:1–45, 1988.

[153] I. Raeburn and J. Taylor. Continuous-trace C^*-algebras with given Dixmier-Douady class. *J. Austral. Math. Soc. (Series A)*, 38:394–407, 1985.

[154] I. Raeburn and D. Williams. Pull-backs of C^*-algebras and crossed products by certain diagonal actions. *Trans. Amer. Math. Soc.*, 287:755–777, 1985.

[155] I. Raeburn and D. Williams. Dixmier-Douady classes of dynamical systems and crossed products. *Canad. J. Math.*, 45:1032–1066, 1993.

[156] I. Raeburn and D. Williams. *Morita Equivalence and Continuous Trace C^*-algebras.* AMS Surveys and Monographs. Amer. Math. Soc., Providence, 1998.

[157] A. Ramsay. Virtual groups and group actions. *Adv. in Math.*, 6:253–322, 1971.

[158] A. Ramsay. Boolean duals of virtual groups. *J. Functional Anal.*, 15:56–101, 1974.

[159] A. Ramsay. Nontransitive quasiorbits in Mackey's analysis of group extensions. *Acta Math.*, 137:17–48, 1976.

[160] A. Ramsay. Subojects of virtual groups. *Pacific J. Math.*, 87:389–454, 1980.

[161] A. Ramsay. Topologies for measured groupoids. *J. Functional Anal.*, 47:314–343, 1982.

[162] A. Ramsay. Measurable group actions are essentially Borel actions. *Israel J. Math.*, 51:339–346, 1985.

[163] A. Ramsay. The Mackey-Glimm dichotomy for foliations and other Polish groupoids. *J. Funct. Anal.*, 94:358–374, 1990.

[164] A. Ramsay. Local product structure for group actions. *Ergodic Theory and Dynamical Systems*, 11:209–217, 1991.

[165] A. Ramsay. Lacunary sections for locally compact groupoids. *Ergodic Theory Dynam. Systems*, 17:933–940, 1997.

[166] A. Ramsay and M. Walter. Fourier-Stieltjes algebras of locally compact groupoids. In R. Doran, editor, *Selfadjoint and Nonselfadjoint Operator Algebras and Operator Theory (Fort Worth, TX, 1990)*, Contemp. Math. 120, pages 143–156, Providence, RI, 1991. Amer. Math. Soc.

[167] A. Ramsay and M. Walter. Noncommutative harmonic analysis on groupoids. In *Topics in Operator Theory, Operator Algebras and Applications (Timişoara, 1994)*, pages 251–264, Bucharest, 1995. Rom. Acad.

[168] A. Ramsay and M. Walter. Fourier-Stieltjes algebras of locally compact groupoids. *J. Funct. Anal.*, 148:314–367, 1997.

[169] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I. Functional Analysis.* Academic Press, San Diego, 1981. 2nd Printing.

[170] B. Reinhart. *Differential Geometry of Foliations*, volume 99 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, 1983.

[171] J. Renault. *A Groupoid Approach to C^*-Algebras*, volume 793 of *Lecture Notes in Mathematics.* Springer-Verlag, New York-Heidelberg, 1980.

[172] J. Renault. *C^*-algebras of groupoids and foliations*, volume 38 of *Proc. Sympos. Pure Math.*, pages 339–350. Amer. Math. Soc., Providence, 1982.

[173] J. Renault. *Two applications of the dual groupoid of a C^*-algebra*, volume 1132 of *Lecture Notes in Mathematics*, pages 434–445. Springer-Verlag, New York, 1983.

[174] J. Renault. Représentation des produits croisés d'algèbres de groupoïdes. *J. Operator Theory*, 18:67–97, 1987.

[175] J. Renault. The ideal structure of groupoid crossed product C^*-algebras. *J. Operator Theory*, 25:3–36, 1991.

[176] J. Renault. Multiplicateurs de Fourier et fonctions de Littlewood pour les groupoïdes r-discrets. *C. R. Acad. Sci. Paris Sér. I Math.*, 319:15–19, 1994.

[177] M. Rieffel. Induced representations of C^*-algebras. *Adv. Math.*, 13:176–257, 1974.

[178] M. Rieffel. *Applications of strong Morita equivalence to transformation group C^*-algebras*, volume 38 of *Proc. Sympos. Pure Math.*, pages 299–310. Amer. Math. Soc., 1982.

[179] J. Roe. Finite propogation speed and Connes' foliation algebra. *Math. Proc. Cambridge Phil. Soc.*, 102:459–466, 1987.

[180] V. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Transl.*, 71:55pp, 1952.

[181] W. Rudin. *Functional Analysis.* McGraw–Hill Book Co., New York, 1973.

[182] A. Seda. A continuity property for Haar systems of measures. . *Ann. Soc. Sci. Bruxelles Sér. I*, 89:429–433, 1975.

[183] A. Seda. An extension theorem for transformation groupoids. *Proc. Roy. Irish Acad. Sect. A*, 75:255–262, 1975.

[184] A. Seda. On compact transformation groupoids. *Cahiers Topologie Géom. Différentielle*, 16:409–414, 1975.

[185] A. Seda. Transformation groupoids and bundles of Banach spaces. *Cahiers Topologie Géom. Différentielle*, 19:131–146, 1978.

[186] A. Seda. Quelques résultats dans la catégorie des groupoïdes d'opérateurs. *C. R. Acad. Sci. Paris Sér. A-B*, 288:A21–A24, 1979.

[187] A. Seda. On measures in fibre spaces. *Cahiers Topologie Géom. Différentielle*, 21:247–276, 1980.

[188] A. Seda. Banach bundles and a theorem of J. M. G. Fell. *Proc. Amer. Math. Soc.*, 83:812–816, 1981.

[189] A. Seda. Banach bundles of continuous functions and an integral representation theorem. *Trans. Amer. Math. Soc.*, 270:327–332, 1982.

[190] A. Seda. On the categories $\mathrm{Sp}(X)$ and $\mathrm{Ban}(X)$. *Cahiers Topologie Géom. Différentielle*, 24:97–112, 1983.

[191] A. Seda. Sur les espaces de fonctions et les espaces de sections. *C. R. Acad. Sci. Paris Sér. I Math.*, 297:41–44, 1983.

[192] A. Seda. Integral representation of linear functionals on spaces of sections. *Proc. Amer. Math. Soc.*, 91:549–555, 1984.

[193] A. Seda. One aspect of the work of Alain Connes. *Irish Math. Soc. Newslett.*, 11:38–48, 1984.

[194] A. Seda. On the categories $\mathrm{Sp}(X)$ and $\mathrm{Ban}(X)$. II. *Cahiers Topologie Géom. Différentielle Catégoriques*, 26:121–133, 1985.

[195] A. Seda. On the continuity of Haar measure on topological groupoids. *Proc. Amer. Math. Soc.*, 96:115–120, 1986.

[196] A. Seda and A. Wickstead. Haar measures for groupoids. *Proc. Roy. Irish Acad. Sect. A*, 76:25–36, 1976.

[197] I. Vaisman. *Lectures on the Geometry of Poisson Manifolds*, volume 118 of *Progress in Mathematics*. Birkhäuser, Boston, 1994.

[198] A. Weinstein. Blowing up realizations of Heisenberg-Poisson manifolds. *Bull. Sci. Math.*, 113:381–406, 1989.

[199] A. Weinstein. Groupoids: Unifying internal and external symmetry. *Notices of the Amer. Math. Soc.*, 43:744–752, 1996.

[200] J. Westman. Locally trivial C^r groupoids and their representations. *Pacific J. Math.*, 20:339–349, 1967.

[201] J. Westman. Nontransitive groupoid algebras. Unpublished preprint from the University of California, Irvine, 1967.

[202] J. Westman. Groupoid theory in algebra, topology, and analysis. Unpublished preprint from the University of California at Irvine., 1971.

[203] D. Williams. The topology on the primitive ideal space of transformation group C^*-algebras and C.C.R. transformation group C^*-algebras. *Trans. Amer. Math. Soc.*, 266:335–359, 1981.

[204] H. Winkelnkemper. The graph of a foliation. *Ann. Glob. Analysis and Geometry*, 1:51–75, 1983.

[205] R. Zimmer. *Ergodic Theory and Semisimple Groups.* Birkhäuser, Boston, 1984.

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