A short introduction to graph groupoids and their algebras

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1. Introduction

This is the contents of the slides I used when I gave 5 lectures at the 2nd International Workshop on Leavitt path algebras and graph C^*-algebras at Kharazmi University, Tehran, Iran, 8–10 July 2019.

I thank the organisers, Mehdi Aaghabali, Massoud Amini, Mohammad Bagher Asadi, Amirhossein Sanatpour, and Zakeieh Tajfirouz, very much for inviting me to this wonderful workshop and for giving me the oppertunity to give the participants of the workshop an introduction to graph groupoids and their algebras.

Compared to the original slides, I have added a few remarks, some references, and a few proofs.

These notes are very incomplete, and that are doubtlessly many mistakes in them. The references are also very sporadic, and I apologise in advance to anyone whose work I have wrongfully neglected to cite.

2. Étale groupoids

2.1. **The definition of a groupoid.**

Definition 2.1. A *groupoid* is a small category in which every morphism has an inverse.

If G is a groupoid, then we write

- $G^{(0)}$ for the set of objects, and
- $G^{(1)}$ for the set of morphisms.

Usually, we will identify an object with its corresponding identity morphism and just write G instead of $G^{(1)}$.

For a morphism $\eta \in G$ we write

- $s(\eta)$ for its domain or source,
- $r(\eta)$ for its codomain or range.
- η^{-1} for its inverse.
- We thus have maps $r, s : G \to G^{(0)}$ and a map $\eta \mapsto \eta^{-1}$ from G to G.

- The composition or product $\eta_1\eta_2$ of $\eta_1, \eta_2 \in G$ is then defined if and only if $s(\eta_1) = r(\eta_2)$.
- We let $G^{(2)} := \{(\eta_1, \eta_2) \in G \times G : s(\eta_1) = r(\eta_2)\}$.

The sets G, $G^{(0)}$, and $G^{(2)}$, and the maps $r, s : G \to G^{(0)}$, $\eta \mapsto \eta^{-1}$, and $(\eta_1, \eta_2) \mapsto \eta_1\eta_2$ have the following properties (see for example [Sims, 2017, Section 2]).

(1) $r(x) = x = s(x)$ for all $x \in G^{(0)}$.

(2) $r(\eta)\eta = \eta = \eta s(\eta)$ for all $\eta \in G$.

(3) $r(\eta^{-1}) = s(\eta)$ and $s(\eta^{-1}) = r(\eta)$ for all $\eta \in G$.

(4) $\eta^{-1}\eta = s(\eta)$ and $\eta\eta^{-1} = r(\eta)$ for all $\eta \in G$.

(5) $r(\eta_1\eta_2) = r(\eta_1)$ and $s(\eta_1\eta_2) = s(\eta_2)$ for all $(\eta_1, \eta_2) \in G^{(2)}$.

(6) $(\eta_1\eta_2)\eta_3 = \eta_1(\eta_2\eta_3)$ whenever $(\eta_1, \eta_2), (\eta_2, \eta_3) \in G^{(2)}$.

Conversely; if G is a set, $G^{(0)}$ is a subset of G, r and s are maps from G to $G^{(0)}$, $G^{(2)} := \{(\eta_1, \eta_2) \in G \times G : s(\eta_1) = r(\eta_2)\}$, and $(\eta_1, \eta_2) \mapsto \eta_1\eta_2$ is a map from $G^{(2)}$ to G, and $\eta \mapsto \eta^{-1}$ is a map from G to G such that (1)–(6) hold; then there is a groupoid such that G is its set of morphism, $G^{(0)}$ is its set of objects, $s(\eta)$ is the domain, $r(\eta)$ is the codomain, and η^{-1} is the inverse of an morphisms η, and the composition of two morphisms η_1 and η_2 for which $s(\eta_1) = r(\eta_2)$ is $\eta_1\eta_2$ (see [Sims, 2017, Remark 2.1.5]).

Thus, instead of Definition 2.1, an alternative, but equivalent way of defining a groupoid is to say that a groupoid consists of a set G, a subset $G^{(0)}$ of G, maps r and s from G to $G^{(0)}$, a map $(\eta_1, \eta_2) \mapsto \eta_1\eta_2$ from $\{(\eta_1, \eta_2) \in G \times G : s(\eta_1) = r(\eta_2)\}$ to G, and a map $\eta \mapsto \eta^{-1}$ from G to G such that (1)–(6) hold.

There is an alternative way of characterising a groupoid where one specify the set G, the set $G^{(2)}$, the product, and the inverse, and then define the set $G^{(0)}$ and the maps r and s from this (see [Renault, 1980, Definition 1.1.1] or [Hahn, 1978, Definition 1.1]).

2.2. **Examples of groupoids.**

(1) Let G be a group and let e be its identity. Then G is a groupoid with $G^{(0)} := \{e\}$, and the product and inverse given by the group operations.

(2) Let X be a set. Then X is a groupoid with $X^{(0)} := X$, r and s the identity maps, the product defined by $(x, x) \mapsto x$, and the inverse defined by $x^{-1} = x$.

(3) Let (E, X, π) be a group bundle, i.e., E and X are sets, π is a surjective map from E to X, and $\pi^{-1}(x)$ is a group for each $x \in X$. Then E is

a groupoid with $E^{(0)} = \{e_x : x \in X\}$, where for each $x \in X$, e_x is the identity of $\pi^{-1}(x)$; $r(\eta) = s(\eta) = e_{\pi(\eta)}$ and η^{-1} is the inverse of η in $\pi^{-1}(\pi(\eta))$; and the product of η_1 and η_2 is the product of η_1 and η_2 in $\pi^{-1}(\pi(\eta_1)) = \pi^{-1}(\pi(\eta_2))$.

(4) Let X be a set and $\sim$ an equivalence relation on X. Let $G := \{(x,y) \in X \times X : x \sim y\}$, let $G^{(0)} := \{(x,x) \in G : x \in X\}$ which we identify with X, and define $r, s : G \to X$ by $r(x,y) = x$ and $s(x,y) = y$. For $(x_1,y_1),(x_2,y_2)$ with $y_1 = x_2$, let $(x_1,y_1)(x_2,y_2) = (x_1,y_2)$; and let $(x,y)^{-1} = (y,x)$ for $(x,y) \in G$. Then G is a groupoid.

(5) Let Γ be a group acting on the right on a set X. We write $x\gamma$ for the action of γ on x. Let

$$X \rtimes \Gamma := X \times \Gamma.$$

Let $(X \rtimes \Gamma)^{(0)} := X \times \{e\}$, which we identify with X, and define $r, s : X \rtimes \Gamma \to X$ by $r(x,\gamma) = x$ and $s(x,\gamma) = x\gamma$. Then $\big((x_1,\gamma_1),(x_2,\gamma_2)\big) \in (X \rtimes \Gamma)^{(2)}$ if and only if $\gamma_2 = x_1\gamma_1$, in which case we let $(x_1,\gamma_1)(x_1\gamma_1,\gamma_2) := (x_1,\gamma_1\gamma_2)$. We also let $(x,\gamma)^{-1} := (x\gamma,\gamma^{-1})$. Then $X \rtimes \Gamma$ is a groupoid.

2.3. **Isotropy and orbits.** Let G be a groupoid. For $x \in G^{(0)}$ let

- $xG := G^x := \{\eta \in G : r(\eta) = x\}$,
- $Gx := G_x := \{\eta \in G : s(\eta) = x\}$,
- $xGx := G_x^x := xG \cap Gx = \{\eta \in G : s(\eta) = r(\eta) = x\}$.

Let $\mathrm{Iso}(G) := \bigcup_{x \in G^{(0)}} xGx = \{\eta \in G : s(\eta) = r(\eta)\}$. The groupoid G is *principal* if $\mathrm{Iso}(G) = G^{(0)}$.

- The *orbit* of an $x \in G^{(0)}$ is the set $\{r(\eta) : \eta \in Gx\}$.
- If $r(\eta) = x$ and $s(\eta) = x'$, then $\eta' \mapsto \eta\eta'\eta^{-1}$ is an isomorphism from $x'Gx'$ to xGx.

The groupoid G is *transitive* if $\mathrm{orb}(x) = G^{(0)}$ for some, and thus for all, $x \in G^{(0)}$.

2.4. **Invariant and full subsets, and bisections.**

- If $U \subseteq G^{(0)}$, then we let $GU := \{\eta \in G : s(\eta) \in U\}$, $UG := \{\eta \in G : r(\eta) \in U\}$, and $G|_U := UG \cap GU$.
- Then $G|_U$ is a subgroupoid of G.
- We say that U is *invariant* if $UG = GU$,
- and that U is *full* if $r(GU) = G^{(0)}$.

- A subset A of a groupoid G is called a *bisection* if the restrictions of r and s to A are both injective.

2.5. **Topological groupoids.**

Definition 2.2. A *topological groupoid* is a groupoid G endowed with a topology under which the maps r are s are continous maps from G to $G^{(0)}$, the map $\eta \mapsto \eta^{-1}$ is a continous map from G to G, and the map $(\eta_1, \eta_2) \mapsto \eta_1\eta_2$ is a continous map from $G^{(2)}$ to G.

2.6. **Examples of topological groupoids.**

(1) If G is any groupoid, then G becomes a topological groupoid if we equip it with the discrete topology.

(2) Let Γ be a topological group acting continuously on the right on a topological space X. Then $X \rtimes \Gamma$ is a topological groupoid if we endow $X \rtimes \Gamma = X \times \Gamma$ with the product topology.

2.7. **Minimal groupoids.** A topological groupoid G is *minimal* if $\emptyset$ and $G^{(0)}$ are the only invariant open subsets of $G^{(0)}$.

Since the open invariant subsets of the unit space of the transformation groupoid of a continuous group action on a topological space X bijectively correspond to the open subsets of X invariant under the action, the transformation groupoid of a continuous group action on a topological space is minimal if and only if the action is minimal.

In general, a topological groupoid G is minimal if and only if $\operatorname{orb}(x)$ is dense in $G^{(0)}$ for every $x \in G^{(0)}$.

2.8. **Hausdorff groupoids.** In general, neither G nor $G^{(0)}$ need to be Hausdorff.

Lemma 2.3. *If G is a topological groupoid, then $G^{(0)}$ is Hausdorff if and only if $G^{(2)}$ is closed in the product topology of $G \times G$.*

Proof. Suppose first that $G^{(0)}$ is Hausdorff and let $(\eta_i, \zeta_i)_{i\in I}$ be a net of elements from $G^{(0)}$ converging to (η, ζ) in $G \times G$. Then $s(\eta_i) = r(\zeta_i)$ for each $i \in I$. It follows from the continuity of r and s that $s(\eta) = \lim s(\eta_i)$ and $r(\zeta) = \lim r(\zeta_i)$. Since $G^{(0)}$ is Hausdorff and $s(\eta_i) = r(\zeta_i)$ for each $i \in I$, we conclude that $s(\eta) = \lim s(\eta_i) = \lim r(\zeta_i) = r(\zeta)$ and thus $(\eta, \zeta) \in G^{(2)}$. This shows that $G^{(2)}$ is closed in $G \times G$.

Suppose then that $G^{(2)}$ is closed in $G \times G$. To show that $G^{(0)}$ is Hausdorff is suffices to prove that limits in $G^{(0)}$ are unique. So suppose that $(x_i)_{i\in I}$ is a net of elements from $G^{(0)}$ that converges to both x and x' in $G^{(0)}$. □

The following lemma is taken from [Sims, 2017].

Lemma 2.4. *If G is a topological groupoid, then $G^{(0)}$ is closed in G if and only if G is Hausdorff.*

Proof. Suppose first that G is Hausdorff and let $(x_i)_{i\in I}$ be a net of elements from $G^{(0)}$ that converges to x in G. Since r is continuous, it follows that $x_i = r(x_i) \to r(x)$. Since G is Hausdorff the limit of $(x_i)_{i\in I}$ is unique, so $x = r(x) \in G^{(0)}$. This shows that $G^{(0)}$ is closed in G.

Suppose then that $G^{(0)}$ is closed in G. To show that G is Hausdorff is suffices to show that limits of nets in G are unique. So suppose $(\eta_i)_{i\in I}$ is a net of elements from G and that $x_i \to \alpha$ and $x_i \to \beta$ in G. Then $x_i^{-1}x_i \to \alpha^{-1}\beta$. Since $x_i^{-1}x_i = s(x_i) \in G^{(0)}$ and $G^{(0)}$ is closed in G, it follows that $\alpha^{-1}\beta \in G^{(0)}$ and thus that $\alpha = \beta$. This shows that limits of nets in G are unique and thus that G is Hausdorff. $\square$

In these notes, most of the topological groupoids we consider are Hausdorff, but there are plenty of natural occuring examples of topological groupoids that are not Hausdorff.

2.9. **Semi-étale groupoids.** A continous map $\phi : X \to Y$ between topological spaces is said to be *locally injective* if every $x \in X$ has an open neighbourhood U such that $\phi|_U$ is injective.

Proposition 2.5 (Cf. [Thomsen, 2010, Lemma 2.1])**.** *Let G be a topological groupoid. Then the following are equivalent.*

(1) The map $r : G \to G^{(0)}$ is locally injective.

(2) The map $s : G \to G^{(0)}$ is locally injective.

(3) The topology on G has a basis consting of open bisections.

(4) $G^{(0)}$ is open in G.

A topological groupoid satisfying the above conditions is said to be *semi-étale* (see [Thomsen, 2010]) or *r-discrete* (see [Renault, 1980]).

Lemma 2.6 (Cf. [Renault, 1980, I.2.7] and [Thomsen, 2010, Lemma 2.2])**.** *Let G be an r-discrete groupoid and let $x \in G^{(0)}$. Then xG and Gx are both discrete subsets of G.*

2.10. **Étale groupoids.** A continous map $\phi : X \to Y$ between topological spaces is called a *local homeomorphism* if every $x \in X$ has an open neighbourhood U such that $\phi(U)$ is open in Y and $\phi|_U : U \to \phi(U)$ is a homeomorphism. Then ϕ is a local homeomorphism if and only if it is locally injective and open.

Proposition 2.7. *Let G be a topological groupoid. Then the following are equivalent.*

(1) The map $r : G \to G$ is a local homeomorphism.

(2) The map $s : G \to G$ is a local homeomorphism.

(3) The map $r : G \to G^{(0)}$ is a local homeomorphism.

(4) The map $s : G \to G^{(0)}$ is a local homeomorphism.

(5) $G^{(0)}$ is open in G and the map $r : G \to G^{(0)}$ is open.

(6) $G^{(0)}$ is open in G and the map $s : G \to G^{(0)}$ is open.

(7) The map $r : G \to G$ is open.

(8) The map $s : G \to G$ is open.

Proof. The equivalences 1. $\Leftrightarrow$ 2., 3. $\Leftrightarrow$ 4, 5. $\Leftrightarrow$ 6. and 7. $\Leftrightarrow$ 8. all follows from the facts that $s(\eta) = r(\eta^{-1})$ for every $\eta \in G$ and $\eta \mapsto \eta^{-1}$ is a homeomorphism.

1. $\Rightarrow$ 3.: It is obvious that if $r : G \to G$ is a local homeomorphism, then $r : G \to G^{(0)}$ is a local homeomorphism.

3. $\Rightarrow$ 5.: It follows from Proposition 2.5 that if $r : G \to G^{(0)}$ is a local homeomorphism, then $G^{(0)}$ is open in G; and if $r : G \to G^{(0)}$ is a local homeomorphism, then it is open.

5. $\Rightarrow$ 7.: It is obvious that if $G^{(0)}$ is open in G and the map $r : G \to G^{(0)}$ is open, then $r : G \to G$ is open.

7. $\Rightarrow$ 1.: If $r : G \to G$ is open, then $G^{(0)} = r(G)$ is open, and it follows from Proposition 2.5 that $r : G \to G^{(0)}$ is locally injective. If $r : G \to G$ is open, then $r : G \to G^{(0)}$ is open, and thus a local homeomorphism. □

A topological groupoid satisfying the above conditions is said to be *étale*.

2.11. **Examples of étale groupoids.**

Example 2.8. If G is any groupoid, then G becomes an étale groupoid if we equip it with the discrete topology.

Example 2.9. Let Γ be a topological group acting continuously on the right on a topological space X. Then the transformation groupoid $X \rtimes \Gamma$ is étale if and only if it is semi-étale, and if and only if Γ is discrete.

Proof. Let $x \in X$. Then $\gamma \mapsto (x, \gamma)$ is a homeomorphism from Γ to $x(X \rtimes \Gamma)$. It therefore follows from Lemma 2.6 that if $X \rtimes \Gamma$ is semi-étale, then Γ is discrete.

Suppose Γ is discrete, and let U be an open subset of $X \rtimes \Gamma$. Then $r(U) = \{(x, e) : (x, \gamma) \in U \text{ for some } \gamma \in \Gamma\}$ is open. This shows that the map $r : G \to G$ is open and thus that $X \rtimes \Gamma$ is étale.

Finally, if $X \rtimes \Gamma$ is étale, then it is also semi-étale. $\square$

2.12. **Deaconu–Renault groupoids.** (See for example [Deaconu, 1995], [Thomsen, 2010, Section 4], [A-D, 1997, Example 1.2(c)], [Renault, 1980, III.2], [Renault, 2000, Section 2], and [Sims&Williams, 2016, Section 3]).

Let X be a locally compact Hausdorff space, let M be a submonoid of an abelian group Γ, and let $(U_m, \sigma_m)_{m \in M}$ be a family of pairs such that each U_m is an open subset of X and each $\sigma_m : U_m \to X$ is a continuous locally injective map such that $U_0 = X$ and $\sigma_0 = \mathrm{id}_X$; and such that if $m, n \in M$, then there is a $k \in M$ such that $k - m, k - n \in M$ and $U_m \cap U_n \subseteq U_k$; and $x \in U_{m+n}$ if and only if $x \in U_n$ and $\sigma_n(x) \in U_m$, in which case $\sigma_m(\sigma_n(x)) = \sigma_{m+n}(x)$.

Let

$$G(X, (U_m, \sigma_m)_{m \in M}) := \bigcup_{m,n \in M} \{(x, m - n, y) : x \in U_m,\ y \in U_n,\ \sigma_m(x) = \sigma_n(y)\}$$

let $G(X, (U_m, \sigma_m)_{m \in M})^{(0)} := \{(x, 0, x) : x \in X\}$ which we identify with X in the obvious way, and define $r, s : G(X, (U_m, \sigma_m)_{m \in M}) \to X$ by $r(x, k, y) = x$ and $s(x, k, y) = y$.

For (x_1, k_1, y_1), $(x_2, k_2, y_2) \in G(X, (U_m, \sigma_m)_{m \in M})$ with $y_1 = x_2$, let

$$(x_1, k_1, y_1)(x_2, k_2, y_2) = (x_1, k_1 + k_2, y_2) \in G(X, (U_m, \sigma_m)_{m \in M});$$

and for $(x, k, y) \in G(X, (U_m, \sigma_m)_{m \in M})$, let

$$(x, k, y)^{-1} := (y, -k, x) \in G(X, (U_m, \sigma_m)_{m \in M}).$$

Then $G(X, (U_m, \sigma_m)_{m \in M})$ is a groupoid.

For $m, n \in M$ and open subsets A of U_m and B of U_n for which $(\sigma_m)|_A$ and $(\sigma_n)|_B$ are injective, let

$$Z(A, m, n, B) := \{(x, m - n, y) : x \in A,\ y \in B,\ \sigma_m(x) = \sigma_n(y)\}.$$

Then the collection

$$\begin{aligned}\{Z(A, m, n, B) : & m, n \in M,\ A \text{ is an open subset of } U_m, \\ & B \text{ is an open subset of } U_n,\ (\sigma_m)|_A \text{ and } (\sigma_n)|_B \text{ are injective}\}\end{aligned}$$

is a basis for a topology on $G(X, (U_m, \sigma_m)_{m \in M})$ that makes $G(X, (U_m, \sigma_m)_{m \in M})$ a locally compact semi-étale Hausdorff groupoid called the *Deaconu–Renault groupoid* of $(X, (U_m, \sigma_m)_{m \in M})$. If each $\sigma_m : U_m \to X$ is a local homeomorphism, then $G(X, (U_m, \sigma_m)_{m \in M})$ is étale.

Proof. It is straight forward to check that $G(X,(U_m,\sigma_m)_{m\in M})$ is a groupoid. Since

$$Z(A_1,m_1,n_1,B_1)\cap Z(A_2,m_2,n_2,B_2)=Z(A_1\cap A_2,m,n,B_1\cap B_2)$$

if $m_1-n_1=m_2-n_2$, where $m,n\in M$ are such that $m-m_1,m-m_2,n-n_1,n_2\in M$, $U_{m_1}\cup U_{m_2}\subseteq U_m$ and $U_{n_1}\cup U_{n_2}\subseteq U_n$; and empty otherwise, the collection

$$\begin{aligned}\{Z(A,m,n,B):&m,n\in M,\ A \text{ is an open subset of } U_m,\\ &B \text{ is an open subset of } U_n,\ (\sigma_m)|_A \text{ and } (\sigma_n)|_B \text{ are injective}\}\end{aligned}$$

is a basis for a topology on $G(X,(U_m,\sigma_m)_{m\in M})$. It is easy to see that this topology is Hausdorff.

If $m,n\in M$, A is an open subset of U_m for which $(\sigma_m)|_A$ is injective, B is an open subset of U_n for which $(\sigma_n)|_B$ is injective, K_A is a compact subset of A, and K_B is a compact subset of B, then $K_A\cap(\sigma_m)^{-1}(\sigma_n(K_B))$ is compact, from which it follows that $\{(x,m-n,y):x\in K_A,\ y\in K_B,\ (\sigma_m)(x)=(\sigma_n)(y)\}$ is compact. Since X is locally compact, it follows that $G(X,(U_m,\sigma_m)_{m\in M})$ is locally compact.

It is straight forward to show that $G(X,(U_m,\sigma_m)_{m\in M})$ is a topological groupoid. Since $G(X,(U_m,\sigma_m)_{m\in M})^{(0)}=Z(X,0,0,X)$ is open in $G(X,(U_m,\sigma_m)_{m\in M})$, the groupoid $G(X,(U_m,\sigma_m)_{m\in M})$ is semi-étale.

If each $\sigma_m:U_m\to X$ is a local homeomorphism, $m,n\in M$, A is an open subset of U_m such that $(\sigma_m)|_A:A\to\sigma_m(A)$ is a homeomorphis, B is an open subset of U_n such that $(\sigma_n)|_B:B\to\sigma_n(B)$ is a homeomorphis, and $\sigma_m(A)=\sigma_n(B)$, then $r(Z(A,m,n,B))=A$. It follows that if each $\sigma_m:U_m\to X$ is a local homeomorphism, then $r:G(X,(U_m,\sigma_m)_{m\in M})\to X$ is open and thus that $G(X,(U_m,\sigma_m)_{m\in M})$ is étale. □

2.13. **Locally contracting groupoids.** The following notion was introduced by Anantharaman-Delaroche in [A-D, 1997]. An étale groupoid G is *locally contracting* at $x\in G^{(0)}$ if for every open neighbourhood $V\subseteq G^{(0)}$ of x, there is an open set $W\subseteq V$ and an open bisection $U\subseteq G$ such that $\overline{W}\subseteq s(U)$ and $r(U\overline{W})\subsetneq W$ ($\overline{W}$ is the closure of W in $G^{(0)}$, and $U\overline{W}=\{\eta\in U:s(\eta)\in\overline{W}\}$). The étale groupoid G is *locally contracting* if it is locally contracting at every $x\in G^{(0)}$. It is proven in [Sims, 2017, Lemma 4.4.3] that if G is a minimal étale groupoid, then it is locally contracting at some $x\in G^{(0)}$ if and only if it is locally contracting.

If $G(X,(U_m,\sigma_m)_{m\in M})$ is the Deaconu–Renault groupoid of $(X,(U_m,\sigma_m)_{m\in M})$ and $x\in X$, then $G(X,(U_m,\sigma_m)_{m\in M})$ is locally contracting at x if there for each neighbourhood V of x are and open set $W\subseteq V$, $m,n\in M$, and open sets $A\subseteq U_m$ and $B\subseteq U_n$ such that $(\sigma_m)|_A$ and $(\sigma_n)|_B$ are injective, $\overline{W}\subseteq B\cap\sigma_n^{-1}(\sigma_m(A))$, and $A\cap\sigma_m^{-1}(\sigma_n(B)\cap\overline{W})\subsetneq W$.

2.14. **Effective and topological principal groupoids.**

- If G is an topological groupoid, then we denote by $\mathrm{Iso}(G)^\circ$ the interior of $\mathrm{Iso}(G)$ in G.
- An étale groupoid G is said to be *effective* if $\mathrm{Iso}(G)^\circ = G^{(0)}$,
- and *topologically principal* if $\{x \in G^{(0)} : xGx = \{x\}\}$ is dense in in $G^{(0)}$.

Example 2.10. If $G(X, (U_m, \sigma_m)_{m\in M})$ is the Deaconu–Renault groupoid of $(X, (U_m, \sigma_m)_{m\in M})$, then $G(X, (U_m, \sigma_m)_{m\in M})$ is effective if there is no triple (U, n, m) consisting of a nonempty open subset $U \subseteq X$ and distinct elements $m, n \in M$ such that $U \subseteq U_m \cap U_n$ and $\sigma_m(x) = \sigma_n(x)$ for every $x \in U$; and $G(X, (U_m, \sigma_m)_{m\in M})$ is topologically principal if and only if there is no nonempty open subset $U \subseteq X$ such that there for each $x \in U$ are distinct elements $m_x, n_x \in M$ such that $x \in U_{m_x} \cap U_{n_x}$ and $\sigma_{m_x}(x) = \sigma_{n_x}(x)$.

It is proven in [Renault, 2008, Proposition 3.6] that if G is an étale groupoid, then it is effective if it is Hausdorff and topologically principal; and it is topologically principal, if it is second countable and effective, and $G^{(0)}$ has the Baire property (which for example is the case if $G^{(0)}$ is locally compact).

2.15. **AF groupoids.** (See [Giordano et al., 2004, Definition 3.7], [Matui, 2012, Section 2], [Nyland&Ortega, 2019, Section 11.5]).

An étale groupoid is said to be an *AF-groupoid* (*approximately finite dimensional-groupoid*) if $G^{(0)}$ is second countable, locally compact and Hausdorff, and there is an increasing sequence $K_1 \subseteq K_2 \subseteq \cdots \subseteq G$ of subgroupoids such that

(1) each K_n is principal,

(2) $K_n^{(0)} = G^{(0)}$ and $K_n \setminus G^{(0)}$ is compact for each n,

(3) and $\bigcup_{n=1}^{\infty} K_n = G$.

Example 2.11. Let $X = \{0,1\}^{\mathbb{N}}$ and equip it with the product topology. Then X is second countable, compact and Hausdorff. Define an equivalence relation $\sim$ on X by $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \Leftrightarrow$ there exists an $N \in \mathbb{N}$ such that $x_n = y_n$ for $n \geq N$, and let G be the groupoid of $\sim$. For $N \in \mathbb{N}$, let $K_N = \{((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) : x_n = y_n \text{ for } n \geq N\}$ and endow K_N with the relative topology of $X \times X$. Endow G with the topology in which $U \subseteq G$ is open if and only if $U \cap K_N$ is open in K_N for each N. Then G is an étale groupoid.

Since $K_1 \subseteq K_2 \subseteq \cdots \subseteq G$ satisfies the 3 condition above, G is AF.

2.16. **Homology of étale groupoids.** (See [Crainic&Moerdijk, 2000] and [Matui, 2012, Section 3]).

If $\phi : X \to Y$ is a continuous local homeomorphism between locally compact Hausdorff spaces, then we define $\phi_* : C_c(X, \mathbb{Z}) \to C_c(Y, \mathbb{Z})$ by $\phi_*(f)(y) = \sum_{\phi(x)=y} f(x)$.

Let G be a locally compact Hausdorff étale groupoid. For $n \in \mathbb{N}$, let

$$G^{(n)} := \{(\eta_1, \ldots, \eta_n) : s(\eta_i) = r(\eta_{i+1}) \text{ for } i = 1, \ldots, n-1\}.$$

For $i = 0, 1, \ldots, n$, define $d_i : G^{(n)} \to G^{(n-1)}$ by

$$d_i((\eta_1, \ldots, \eta_n)) = \begin{cases} (\eta_2, \ldots, \eta_n) & \text{if } i = 0, \\ (\eta_1, \ldots, \eta_i\eta_{i+1}, \eta_n) & \text{if } 0 < i < n, \\ (\eta_1, \ldots, \eta_{n-1}) & \text{if } i = n. \end{cases}$$

Then d_i is a continuous local homeomorphism.

Define $\delta_n : C_c(G^{(n)}, \mathbb{Z}) \to C_c(G^{(n-1)}, \mathbb{Z})$ by $\delta_1 = s_* - r_*$ and $\delta_n = \sum_{i=0}^{n} (-1)^i d_{i*}$. Then

$$0 \xleftarrow{\delta_0} C_c(G^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(G^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(G^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \ldots$$

is a chain complex. The nth homology group of G is $H_n(G) := \ker \delta_n / \operatorname{im} \delta_{n+1}$.

Example 2.12. Let Γ be a discrete group that acts continuously on the right on a locally compact Hausdorff space X. Then $\operatorname{im} \delta_1 = \{f - f(\cdot\gamma) : f \in C_c(X, \mathbb{Z}),\ \gamma \in \Gamma\}$. So $H_0(X \rtimes \Gamma) = C_c(X, \mathbb{Z}) / \{f - f(\cdot\gamma) : f \in C_c(X, \mathbb{Z}),\ \gamma \in \Gamma\}$.

3. Graph groupoids

In this section we construct a groupoid $G(E)$ from an arbitrary graph E and look at some properties of this groupoid. The groupoid $G(E)$ was first introduced in [Kumjian et al., 1997] for locally finite graphs. Later, Paterson constructed in [Paterson, 2002] $G(E)$ for graphs with no sinks by using inverse semigroups. The description of $G(E)$ given here is a speciale case of the boundary path space constructed from topological higher-rank graphs in [Yeend, 2007], and can be found in for example [Clark&Sims, 2015] and [Brownlowe et al., 2017].

3.1. **The boundary path space of a graph.** The *boundary path space* of a graph was constructed in [Webster, 2014].

Let $E = (E^0, E^1, r, s)$ be a graph.

- We let $E^0_{\text{reg}} := \{v \in E^0 : vE^1 \text{ is finite and nonempty}\}$ and $E^0_{\text{sing}} := E^0 \setminus E^0_{\text{reg}}$.

- A *finite path* in E is an finite sequence $e_1e_2\dots e_n$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i. The *length* of a finite path $\mu = e_1e_2\dots e_n$ is $|\mu| := n$. We let E^n be all the paths of length n, and let $E^* = \bigcup_{n=0}^{\infty} E^n$. The source and range maps extends to E^* in the obvious way.
- An *infinite path* in E is an infinite sequence $x_1x_2\dots$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i. We let E^∞ be the set of all infinite paths in E. The source map extends to E^∞ in the obvious way. We let $|x| = \infty$ for $x \in E^\infty$.
- The *boundary path space* of E is the space
$$\partial E := E^\infty \cup \{\mu \in E^* : r(\mu) \in E^0_{\mathrm{sing}}\}.$$
- If $\mu = \mu_1\mu_2\cdots\mu_m \in E^*$, $x = x_1x_2\cdots \in E^* \cup E^\infty$ and $r(\mu) = s(x)$, then we let μx denote the path $\mu_1\mu_2\cdots\mu_m x_1x_2\cdots \in E^* \cup E^\infty$.
- For $\mu \in E^*$, the *cylinder set* of μ is the set
$$Z(\mu) := \{\mu x \in \partial E : x \in r(\mu)\partial E\},$$
where $r(\mu)\partial E := \{x \in \partial E : r(\mu) = s(x)\}$.
- Given $\mu \in E^*$ and a finite subset $F \subseteq r(\mu)E^1$ we define
$$Z(\mu \setminus F) := Z(\mu) \setminus \left(\bigcup_{e \in F} Z(\mu e) \right).$$
- ∂E is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F) : \mu \in E^*,\ F \text{ is a finite subset of } r(\mu)E^1\}$, and each such $Z(\mu \setminus F)$ is compact and open.
- For $n \in \mathbb{N}_0$, let $\partial E^{\geq n} := \{x \in \partial E : |x| \geq n\}$.
- Then $\partial E^{\geq n} = \cup_{\mu \in E^n} Z(\mu)$ is an open subset of ∂E.
- For $n \geq 1$, we define the *n-shift map* on E to be the map $\sigma_n : \partial E^{\geq n} \to \partial E$ given by $\sigma_n(x_1x_2x_3\cdots x_nx_{n+1}\dots) = x_{n+1}\cdots$ for $x_1x_2x_3\cdots x_nx_{n+1}\cdots \in \partial E^{\geq n+1}$ and $\sigma_n(\mu) = r(\mu)$ for $\mu \in \partial E \cap E^n$.
- We let σ_0 denote the identity map on ∂E.
- Then $\sigma_n : \partial E^{\geq n} \to \partial E$ is a local homeomorphism for all $n \in \mathbb{N}_0$.
- When we write $\sigma_n(x)$, we implicitly assume that $x \in \partial E^{\geq n}$.

3.2. **Eventually periodic points.**

- A *loop* or a *cycle* is a path $\nu \in E^* \setminus E^0$ such that $r(\nu) = s(\nu)$. A loop ν is said to be *simple* if there is no other loop ν' such that $\nu = \nu'\nu'\dots\nu'$.

- $x \in \partial E$ is said to be *eventually periodic* if there are $m, n \in \mathbb{N}_0$, $m \neq n$ such that $\sigma_m(x) = \sigma_n(x)$. We let
$$\operatorname{per}(x) = \min\{m - n : m, n \in \mathbb{N}_0,\ m > n,\ \sigma_m(x) = \sigma_n(x)\}$$
if x is eventually periodic, and let $\operatorname{per}(x) = 0$ otherwise.
- $x \in \partial E$ is eventually periodic if and only if $x = \mu\nu\nu\nu\cdots$ for some path $\mu \in E^*$ and some loop $\nu \in E^*$ with $s(\nu) = r(\mu)$. By replacing ν by a subloop if necessary, we can assume that ν is a simple loop in which case $\operatorname{per}(x) = |\nu|$.

3.3. **The groupoid of a graph.**

- ∂E is a locally compact Hausdorff space, $\mathbb{N}_0$ is a submonoid of the abelian group $\mathbb{Z}$, and $(\partial E^{\geq n}, \sigma_n)_{n \in \mathbb{N}_0}$ is a family of pairs such that each $\partial E^{\geq n}$ is an open subset of ∂E and each $\sigma_n : \partial E^{\geq E} \to \partial E$ is a continuous local homeomorphism such that $x \in \partial E^{\geq m+n}$ if and only if $x \in \partial E^{\geq n}$ and $\sigma_n(x) \in \partial E^{\geq m}$, in which case $\sigma_m(\sigma_n(x)) = \sigma_{m+n}(x)$.
- We define the groupoid $G(E)$ of E to be the Deaconu–Renault groupoid of $(\partial E, (\partial E^{\geq n}, \sigma_n)_{n \in \mathbb{N}_0})$.
- Then $G(E)$ is a locally compact Hausdorff étale groupoid.
- $G(E) = \{(x, k, y) :$ there exist $m, n \in \mathbb{N}_0$ such that $x \in \partial E^{\geq m},\ y \in \partial E^{\geq n},\ k = m - n,\ \sigma_m(x) = \sigma_n(y)\}$.
- $r((x, k, y)) = x$, $s((x, k, y)) = y$, $(x, k, y)^{-1} = (y, -k, x)$, and $(x, k, y)(y, l, z) = (x, k + l, z)$.
- If $x \in \partial E$, then $\operatorname{orb}(x) = \{y \in \partial E :$ exist $m, n \in \mathbb{N}_0$ such that $\sigma_m(x) = \sigma_n(y)\}$ and $xG(E)x = \{(x, m - n, x) : m, n \in \mathbb{N}_0,\ \sigma_m(x) = \sigma_n(x)\} = \{(x, k\operatorname{per}(x), x) : k \in \mathbb{Z}\}$.
- The topology of $G(E)$ has a basis consisting of compact open sets of the form
$$Z(U, m, n, V) = \{(x, m - n, y) : x \in U,\ y \in V, \sigma_m(x) = \sigma_n(y)\},$$
where $m, n \in \mathbb{N}_0$, U is a compact open subset of $\partial E^{\geq m}$ such that $(\sigma_m)|_U$ is injective, V is a compact open subset of $\partial E^{\geq n}$ such that $(\sigma_n)|_V$ is injective, and $\sigma_m(U) = \sigma_n(V)$.
- We have in particular that sets of the form
$$Z(\mu, \nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu))$$
where $\mu, \nu \in E^*$ and $r(\mu) = r(\nu)$, are compact and open.

3.4. **Open invariant sets.** A subset $A \subseteq E^0$ is said to be *hereditary* if $r(AE^1) \subseteq A$, and *saturated* if $v \in A$ for each $v \in E^0_{\mathrm{reg}}$ for which $r(vE^1) \subseteq A$.

The following result is straight forward to prove.

Proposition 3.1

(1) *If $A \subseteq E^0$ is hereditary and saturated, then $\bigcup_{\mu \in E^*A} Z(\mu)$ is an open invariant subset of ∂E.*

(2) *If U is an open invariant subset of ∂E, then $\{v \in E^0 : Z(v) \subseteq U\}$ is a hereditary and saturated subset of E^0.*

(3) *$A \mapsto \bigcup_{\mu \in E^*A} Z(\mu)$ is a bijection between the hereditary and saturated subset of E^0 and the open invariant subset of ∂E. The inverse of this bijection is the map $U \mapsto \{v \in E^0 : Z(v) \subseteq U\}$.*

3.5. **Cofinal graphs and minimal graph groupoids.** The graph E is said to be *cofinal* if there for every $v, w \in E^0$ is a finite subset $F \subseteq wE^*$ such that $vE^*r(\mu) \neq \emptyset$ for each $\mu \in F$ and $Z(w) = \bigcup_{\mu \in F} Z(\mu)$. Then E is cofinal if and only if there only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

Corollary 3.2. *$G(E)$ is minimal if and only if E is cofinal.*

3.6. **Effective and topological principal graph groupoids.** A loop $\nu = \nu_1 \dots \nu_n \in E^n$ is said to *have an exit* if there is an i such that $r(\nu_i)E^1$ contains at least two elements. The graph E is said to satisfy *Condition (L)* if every loop in E has an exit.

The following is proved in [Nyland&Ortega, 2019, Proposition 8.2].

Proposition 3.3. *The following are equivalent.*

(1) *$G(E)$ is effective.*

(2) *$G(E)$ is topologically principal.*

(3) *The set of boundary paths which are not eventually periodic form a dense subset of the boundary path space ∂E.*

(4) *E satisfies Condition (L).*

3.7. **AF graph groupoids.** The following result is not difficult to prove.

Proposition 3.4. *If E^0 and E^1 are countable, then the following are equivalent.*

(1) *$G(E)$ is AF.*

(2) *$G(E)$ is principal.*

(3) *E contains no loops.*

3.8. Locally contracting graph groupoids.

- $G(E)$ is locally contracting at $x = x_1 x_2 \cdots \in \partial E$ if there for each n is a path $\mu \in s(x_n)E^*$ and a loop $\nu \in r(\mu)E^*$ with an exit.
- $G(E)$ is locally contracting if there for each $v \in E^0$ is a path $\mu \in vE^*$ and a loop $\nu \in r(\mu)E^*$ with an exit.

Corollary 3.5. <4-> *If $G(E)$ is minimal and effective, then $G(E)$ is locally contracting if E contains a loop, and AF otherwise.*

3.9. Homology of graph groupoids.

When A is a set, then we let $\mathbb{Z}^A$ denote the abelian group

$$\{(n_a)_{a\in A} : \text{each } n_a \in \mathbb{Z},\ \text{and } n_a = 0 \text{ for all but finitely many } a\}.$$

For $a_0 \in A$ we let δ_{a_0} be the element $(n_a)_{a\in A} \in \mathbb{Z}^A$ where $n_{a_0} = 1$ and $n_a = 0$ for $a \neq a_0$.

Define a group homomorphism $(1 - A_E^T) : \mathbb{Z}^{E^0_{\text{reg}}} \to \mathbb{Z}^{E^0}$ by

$$(1 - A_E^T)\delta_v = \delta_v - \sum_{e \in vE^1} \delta_{r(e)}.$$

Theorem 3.6 (Cf. [Matui, 2012, Theorem 4.14])

1. *There is an isomorphism from $\mathbb{Z}^{E^0} / \operatorname{im}(1 - A_E^T)$ to $H_0(G(E))$ mapping $[\delta_v]$ to $[1_{Z(v)}]$ for every $v \in E^0$.*
2. *There is an isomorphism from $\ker(1 - A_E^T)$ to $H_1(G(E))$ mapping $(n_v)_{v\in E^0_{\text{reg}}}$ to $\sum_{v\in E^0_{\text{reg}}} n_v \sum_{e\in vE^1} [1_{Z(e,r(e))}]$ for every $(n_v)_{v\in E^0_{\text{reg}}} \in \ker(1 - A_E^T)$.*
3. *$H_n(G(E)) = 0$ for $n \geq 2$.*

4. Groupoid algebras

C^*-algebras of topological groupoids were introduced in [Renault, 1980] and have since then been intensively studied. Many interesting classes of C^*-algebras can be constructed as C^*-algebras of topological groupoids, which therefore provides a unified framework for studying things like the ideal structure, K-theory, KMS-states, pure infiniteness, isomorphisms, and Morita equivalence for these classes of C^*-algebras.

This section contains a short introduction the reduced C^*-algebra and the universal C^*-algebra of an étale Hausdorff groupoid.

I recommend [Renault, 1980], [Paterson, 1999], and [Sims, 2017] for more on groupoids and their C^*-algebras.

4.1. **The convolution algebra.**

- Let G be a locally compact Hausdorff étale groupoid.
- If $f, g \in C_c(G)$ and $\eta \in G$, then the set $\{(\eta_1, \eta_2) \in G^{(2)} : \eta_1\eta_2 = \eta,\ f(\eta_1)g(\eta_2) \neq 0\}$ is finite.
- We can therefore define a function $f * g : G \to \mathbb{C}$ by
$$(f * g)(\eta) := \sum_{\eta_1\eta_2=\eta} f(\eta_1)g(\eta_2).$$
- It is not difficult to check that $f * g \in C_c(G)$.
- The complex vector space $C_c(G)$ is a $*$-algebra with multiplication given by $*$ and involution given by $f^*(\eta) = \overline{f(\eta^{-1})}$.
- $C_c(G) = \mathrm{span}\{f \in C_c(G) : \mathrm{supp}(f) \text{ is a bisection}\}$.

4.2. **The left-regular representation.**

- A *$*$-representation* of $C_c(G)$ on a Hilbert space H is a linear map $\pi : C_c(G) \to B(H)$ such that $\pi(f * g) = \pi(f)\pi(g)$ and $\pi(f^*) = \pi(f)^*$.
- For each $x \in G^{(0)}$ there is a $*$-representation $\pi_x : C_c(G) \to B(l^2(Gx))$ such that $\pi_x(f)\delta_\eta = \sum_{\alpha \in Gr(\eta)} f(\alpha)\delta_{\alpha\eta}$ for $f \in C_c(G)$ and $\eta \in Gx$.
- If $\eta \in G$, then the map $U_\eta : l^2(Gs(\eta)) \to l^2(Gr(\eta))$ given by $U_\eta\delta_\alpha = \delta_{\alpha\eta^{-1}}$ is a unitary operator such that $\pi_{r(\eta)} = U_\eta \pi_{s(\eta)} U_\eta^*$.
- The *left-regular representation* of G is the representation $\pi_r := \oplus_{x \in G^{(0)}} : C_c(G) \to \oplus_{x \in G^{(0)}} B(l^2(Gx))$.

Definition 4.1. The *reduced* C^*-algebra $C_r^*(G)$ of G is the completion of $C_c(G)$ with respect to the norm $||\cdot||_r$ defined by $||f||_r = ||\pi_r(f)||$.

The following result is proved in [Sims, 2017, Proposition 3.3.3].

Proposition 4.2. *There is an injective, norm-decreasing map $j : C_r^*(G) \to C_0(G)$ such that*
$$j(a)(\eta) = \langle \pi_{s(\eta)}(a)\delta_{s(\eta)} | \delta_\eta \rangle$$
for $a \in C_r^*(G)$ and $\eta \in G$. For $f \in C_c(G)$, we have $j(f) = f$.

4.3. **The universal representation.** The following result is proved in [Sims, 2017, Proposition 3.2.1].

Lemma 4.3. *If π is a $*$-representation of $C_c(G)$ and $f \in C_c(G)$ is supported on a bisector, then $||\pi(f)|| \leq ||f||_\infty$.*

Definition 4.4. The *universal* C^*-algebra $C^*(G)$ of G is the completion of $C_c(G)$ with respect to the norm $||\cdot||$ defined by

$$||f|| = \sup\{||\pi(f)|| : \pi \text{ is a } *\text{-representation of } C_c(G)\}.$$

For $f \in C_c(G)$, we have $||f||_\infty \leq ||f||_r \leq ||f||$. If f is supported on a bisector, then $||f||_\infty = ||f||_r = ||f||$, cf. [Sims, 2017, Corollary 3.3.4].

4.4. **Graph C^*-algebras.** We shall now see that the both the universal and the reduced C^*-algebra of $G(E)$ are isomorphic to the C^*-algebra of E, cf. [Kumjian et al., 1997, Proposition 4.1] and [Brownlowe et al., 2017, Proposition 2.2].

- Let $E = (E^0, E^1, r, s)$ be a graph.
- If $\{P_v,\ S_e : v \in E^0,\ e \in E^1\}$ is a Cuntz–Krieger family in a C^*-algebra A, then there is $*$-representation $\pi : C_c(G(E)) \to A$ such that $\pi(1_{Z(\mu,\nu)}) = S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, where $S_\mu = S_{\mu_1} \dots S_{\mu_m}$ if $\mu = \mu_1 \dots \mu_m \in E^m$ for $m \geq 1$, $S_\mu = P_\mu$ if $\mu \in E^0$, $S_\nu = S_{\nu_1} \dots S_{\nu_n}$ if $\nu = \nu_1 \dots \nu_n \in E^n$ for $n \geq 1$, $S_\nu = P_\nu$ if $\nu \in E^0$.
- Conversely, if $\pi : C_c(G) \to A$ is a $*$-representation of $C_c(G)$ on a C^*-algebra A, then $\{\pi(1_{Z(v,v)}),\ \pi(1_{Z(e,r(e))}) : v \in E^0,\ e \in E^1\}$ is a Cuntz–Krieger family.
- It follows that there is a $*$-isomorphism from $C^*(E)$ to $C^*(G(E))$ that for each $v \in E^0$ maps p_v to $1_{Z(v,v)}$, and for each $e \in E^1$ maps s_e to $1_{Z(e,r(e))}$.

4.5. **Amenable groupoids.**

- There is a notion of *amenability* for étale groupoids (see [AnRe, 2000] or [Sims, 2017, Section 4.1]).
- If G is amenable, then the $*$-homomorphism $\pi_r : C^*(G) \to C_r^*(G)$ is injective (see [AnRe, 2000, Proposition 6.1.8] or [Sims, 2017, Theorem 4.1.4]).
- If E is a graph, then $G(E)$ is amenable and $C_r^*(G(E)) = C^*(G(E))$ (see [Yeend, 2007, Proposition 6.2]).

4.6. **Steinberg algebras.**

- If A, B are compact open bisections, then we let $AB := \{\eta_1\eta_2 : \eta_1 \in A,\ \eta_2 \in B,\ s(\eta_1) = r(\eta_2)\}$ and $A^{-1} := \{\eta^{-1} : \eta \in A\}$. Then AB and A^{-1} are both compact open bisections.
- If A and B are compact open bisections, then $ABA = A$ and $BAB = B$ if and only if $A = B^{-1}$.

- An étale groupoid G is *ample* if its topology has a basis consisting of compact open bisections.

The Steinberg algebra of an ample étale groupoid was introduced by Steinberg in [Steinberg, 2010] (see also [Clark et al., 2014]).

Definition 4.5. Let G be an ample étale groupoid, and let R be a unital commutative ring. The *Steinberg algebra* $A_R(G)$ of G with coefficient in R, is the R-algebra $\mathrm{span}_R\{1_A : A \text{ is a compact open bisection}\}$ with multiplication defined by

$$(f * g)(\eta) := \sum_{\eta_1\eta_2=\eta} f(\eta_1)g(\eta_2).$$

4.7. **Leavitt path algebras.** If $E = (E^0, E^1, r, s)$ is a graph and R is a unital commutative ring, then there is an isomorphism from $L_R(E)$ to $A_R(G(E))$ that for each $v \in E^0$ maps v to $1_{Z(v,v)}$, and for each $e \in E^1$ maps e to $1_{Z(e,r(e))}$, see [Clark&Sims, 2015, Example 3.2].

4.8. **Uniqueness theorems.**

- If H is an open subgroupoid of G, then H is locally compact Hausdorff and étale, and the inclusion of $C_c(H)$ into $C_c(G)$ extends to an inclusion of $C_r^*(H)$ into $C_r^*(G)$, see [Phillips, 2005, Proposition 1.9].
- $G^{(0)}$ and $\mathrm{Iso}(G)^\circ$ are open subgroupoids of G, so we can consider $C_r^*(G^{(0)}) = C_0(G^{(0)})$ and $C_r^*(\mathrm{Iso}(G)^\circ)$ to be C^*-subalgebras of $C_r^*(G)$.
- If $\phi : C_r^*(G) \to A$ is a $*$-homomorphism that is injective on $C_r^*(\mathrm{Iso}(G)^\circ)$, then ϕ is injective, see [Brown et al., 2016, Theorem 3.1].
- It follows from the previous result that if G is effective and $\phi : C_r^*(G) \to A$ is a $*$-homomorphism that is injective on $C_0(G^{(0)})$, then ϕ is injective.
- If G is ample, R is a unital commutative ring, and $\pi : A_R(G) \to A$ is a ring homomorphism that is injective on $A_R(\mathrm{Iso}(G)^\circ)$, then π is injective, see [Clark et al., 2016, Theorem 3.1].
- It follows from the previous result that if G is ample and effective, R is a unital commutative ring, and $\pi : A_R(G) \to A$ is a ring homomorphism that is injective on $A_R(G^{(0)})$, then π is injective.

4.9. **Invariant subsets, ideals and quotients.**

- If U is an open invariant subset of $G^{(0)}$, then $G|_U$ is a subgroupoid of G and $C_r^*(G|_U)$ is an ideal in $C_r^*(G)$, see [Sims, 2017, Proposition 4.3.2].

- Moreover, $G^{(0)} \setminus U$ is a closed invariant subset of $G^{(0)}$ and there is a surjective $*$-homomorphism $\pi : C_r^*(G) \to C_r^*(G|_{G^{(0)}\setminus U})$ such that $\pi(f) = f|_{G^{(0)}\setminus U}$ for $f \in C_c(G)$, and
$$0 \to C_r^*(G|_U) \xrightarrow{\iota} C_r^*(G) \xrightarrow{\pi} C_r^*(G|_{G^{(0)}\setminus U}) \to 0$$
is exact if G is amenable, see [Sims, 2017, Proposition 4.3.2].
- We say that G is *strongly effective* if $G|_W$ is effective for all closed invariant subsets W of $G^{(0)}$.
- If G is amenable and strongly effective, then $U \mapsto C_r^*(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the ideals in $C_r^*(G)$, see [Sims, 2017, Theorem 4.3.3].
- If G is ample, R is a unital commutative ring, and U is an open invariant subset of $G^{(0)}$, then there is an exact sequence
$$0 \to A_R(G|_U) \xrightarrow{\iota} A_R(G) \xrightarrow{\pi} A_R(G|_{G^{(0)}\setminus U}) \to 0$$
- If G is ample and strongly effective, and K is a field, then $U \mapsto A_K(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the ideals in $A_K(G)$, [Clark et al., 2016, Theorem 5.4].

4.10. **Cocycles, gradings, and group actions.**

- Let Γ be an abelian group. A *cocycle* from G to Γ is a map $c : G \to \Gamma$ such that $c(\eta^{-1}) = c(\eta)^{-1}$ for $\eta \in G$, and $c(\eta_1\eta_2) = c(\eta_1)c(\eta_2)$ for $(\eta_1, \eta_2) \in G^{(2)}$.
- A continuous cocycle $c : G \to \Gamma$ induces a Γ-*grading* $\{c^{-1}(\gamma)\}_{\gamma\in\Gamma}$ of G (i.e., $\bigcup_{\gamma\in\Gamma} c^{-1}(\gamma) = G$, $c^{-1}(\gamma_1) \cap c^{-1}(\gamma_2) = \emptyset$ for $\gamma_1 \neq \gamma_2$, and $\eta_1\eta_2 \in c^{-1}(\gamma_1\gamma_2)$ if $(\eta_1, \eta_2) \in G^{(2)}$, $\eta_1 \in c^{-1}(\gamma_1)$, and $\eta_2 \in c^{-1}(\gamma_2)$).
- It also induces a Γ-grading $\{A_R^\gamma(G)\}_{\gamma\in\Gamma}$ of the Steinberg algebra $A_R(G)$ of G, where $A_R^\gamma(G) = \{f \in A_R(G) : \operatorname{supp}(f) \subseteq c^{-1}(\gamma)\}$, see for example [Clark et al., 2016, Section 5].
- And a strongly continuous action $\beta^c : \widehat{\Gamma} \to \operatorname{Aut}(C^*(G))$ such that $\beta_\phi^c(f) = \phi(\gamma)f$ for $\phi \in \widehat{\Gamma}$, $\gamma \in \Gamma$ and $f \in C_c(G)$ with $\operatorname{supp}(f) \subseteq c^{-1}(\gamma)$, see for example [Carlsen et al., 2017b, Lemma 6.1].
- The map $(x, k, y) \mapsto k$ is a continuous cocycle from $G(E)$ to $\mathbb{Z}$.
- We thus have a $\mathbb{Z}$-grading $\{G_k(E)\}_{k\in\mathbb{Z}}$ of $G(E)$ where $G_k(E) = \{(x, l, y) \in G(E) : l = k\}$.
- A strongly continuous action $\beta : \mathbb{T} \to \operatorname{Aut}(C^*(E))$ such that $\beta_\gamma(s_\mu s_\nu^*) = \gamma^{|\mu|-|\nu|} s_\mu s_\nu^*$ for $\gamma \in \mathbb{T}$ and $\mu, \nu \in E^*$.

- And a $\mathbb{Z}$-grading $\{L_R^k(E)\}_{k\in\mathbb{Z}}$ of $L_R(E)$ where $L_R^k(E) = \mathrm{span}_R\{\mu\nu^* : \mu,\nu \in E^*,\ |\mu| - |\nu| = k\}$.

4.11. **Invariant and graded ideals.**

- Let Γ be an abelian group and $c : G \to \Gamma$ a continuous cocycle. If $c^{-1}(0)$ is amenable and strongly effective, then $U \mapsto C_r^*(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the set of β^c-invariant ideals in $C_r^*(G)$.
- It follows from the previous result that if $c^{-1}(0)$ is amenable and strongly effective and $\phi : C_r^*(G) \to A$ is a $*$-homomorphism that is injective on $C_0(G^{(0)})$ and for which there is an action $\alpha : \widehat{\Gamma} \to A$ such that $\phi\circ\beta_\zeta^c = \alpha_\zeta\circ\phi$ for all $\zeta \in \widehat{\Gamma}$, then ϕ is injective.
- If G is ample, $c^{-1}(0)$ is strongly effective, and K is a field, then $U \mapsto A_K(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the set of Γ-graded ideals in $A_K(G)$, see [Clark et al., 2016, Theorem 5.4].
- It follows from the previous result that if G is ample, $c^{-1}(0)$ is strongly effective, K is a field, and $\pi : A_R(G) \to A$ is a Γ-graded ring homomorphism that is injective on $A_R(G^{(0)})$, then π is injective.

4.12. **AF and ultramatricial algebras.**

- If G is an AF groupoid, then G is amenable and ample, $C^*(G)$ is an AF algebra and $A_R(G)$ is an ultramatricial algebra, see [Giordano et al., 2004].
- So if E is a graph with no loops, then $C^*(E)$ is an AF algebra and $L_R(E)$ is an ultramatricial algebra, cf. [Kumjian et al., 1998, Theorem 2.4] and [Drinen&Tomforde, 2005, Corollary 2.13].

4.13. **Purely infinite algebras.**

- If G is effective and locally contracting, then $C_r^*(G)$ is purely infinite, see [A-D, 1997, Proposition 2.4] and [Sims, 2017, Theorem 4.4.2].
- So if E is a graph such that every vertex in E connects to a loop with an exit, then $C^*(E)$ is purely infinite, cf. [Bates et al., 2000, Proposition 5.3] and [Drinen&Tomforde, 2005, Corollary 2.14].

5. Orbit equivalence and isomorphism of groupoids

The following result is proven in [Carlsen et al., 2017b, Theorem 3.3].

Theorem 5.1. *Let G_1 and G_2 be locally compact Hausdorff étale groupoids and consider the following two conditions.*

(1) G_1 and G_2 are topological isomorphic.

(2) There is a $*$-isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.

Then $1 \Rightarrow 2$. If moreover G_1 and G_2 are second-countable and each $\mathrm{Iso}(G_i)^\circ$ is torsion-free and abelian, then $2 \Rightarrow 1$.

A ring R is *indecomposable* if 0 and 1 are the only idempotents in R. The following result is a special case of [Steinberg, 2019, Theorem 5.7].

Theorem 5.2. *Let G_1 and G_2 be Hausdorff ample étale groupoids, let R be a unital commutative ring, and consider the following two conditions.*

(1) G_1 and G_2 are topological isomorphic.

(2) There is a R-algebra isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.

(3) There is a ring isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.

Then $1 \Rightarrow 2 \Rightarrow 3$. If moreover each $\mathrm{Iso}(G_i)^\circ$ is free abelian and R is indecomposable, then $3 \Rightarrow 1$.

5.1. **Continuous orbit equivalence.**

- Let X be a second-countable locally compact Hausdorff space and let $(U_n, \sigma_n)_{n\in\mathbb{N}_0}$ be a family of pairs such that each U_n is an open subset of X, $\sigma_n : U_n \to X$ is a local homeomorphism, $U_0 = X$, $\sigma_0 = \mathrm{id}_X$, and $x \in U_{m+n}$ if and only if $x \in U_n$ and $\sigma_n(x) \in U_m$ in which case $\sigma_m(\sigma_n(x)) = \sigma_{m+n}(x)$.
- Recall that $\mathrm{orb}(x) = \{y \in X : \text{there exist } m, n \in \mathbb{N}_0 \text{ such that } \sigma_m(x) = \sigma_n(y)\}$ in $G(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ for $x \in X$.
- It follows that if $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ are two Deaconu–Renault systems and $h : X \to Y$ is a map such that $h(\mathrm{orb}(x)) = \mathrm{orb}(h(x))$ for all $x \in X$, then there are functions $k, l : U_1 \to \mathbb{N}_0$ such that $\tau_{l(x)}(h(x)) = \tau_{k(x)}(h(\sigma_1(x)))$ for $x \in U_1$.

Definition 5.3. Let $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems. Then (h, k, l, k', l') is a *continuous orbit equivalence* if $h : X \to Y$ is a homeomorphism and $k, l : U_1 \to \mathbb{N}_0$ and $k', l' : V_1 \to \mathbb{N}_0$ are continuous maps such that

$$\tau_{l(x)}(h(x)) = \tau_{k(x)}(h(\sigma_1(x)))$$

for $x \in U_1$, and

$$\sigma_{l'(y)}(h^{-1}(y)) = \sigma_{k'(y)}(h^{-1}(\tau_1(y)))$$

for $y \in V_1$.

5.2. **Essential stabiliser-preserving continuous orbit equivalence.**

- If $x \in X$, then $xG(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})x \cap \mathrm{Iso}((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0}))^\circ = \{(x, m-n, x) : m,n \in \mathbb{N}_0,$ there is an open neighbourhood $U \subseteq U_m \cap U_n$ of x such that $\sigma_m(x') = \sigma_n(x')$ for all $x' \in U\}$.
- We say that a continuous orbit equivalence (h,k,l,k',l') between $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ *preserves essential stabilisers* if

$$\begin{aligned}&\left\{\sum_{i=0}^{m-1}(l(\sigma_i(x)) - k(\sigma_i(x))) : m,n \in \mathbb{N}_0, \text{ there is an open neighbourhood}\right.\\ &\qquad\left. U \subseteq U_m \cap U_n \text{ of } x \text{ such that } \sigma_m(x') = \sigma_n(x') \text{ for all } x' \in U\right\}\\ &\qquad = \{m-n : m,n \in \mathbb{N}_0, \text{ there is an open neighbourhood}\\ &\qquad\qquad V \subseteq V_m \cap V_n \text{ of } h(x) \text{ such that } \tau_m(y) = \tau_n(y) \text{ for all } y \in V\}\end{aligned}$$

for all $x \in X$, and

$$\begin{aligned}&\left\{\sum_{i=0}^{m-1}(l'(\tau_i(y)) - k'(\tau_i(y))) : m,n \in \mathbb{N}_0, \text{ there is an open neighbourhood}\right.\\ &\qquad\left. V \subseteq V_m \cap V_n \text{ of } x \text{ such that } \tau_m(y') = \tau_n(y') \text{ for all } y' \in V\right\}\\ &\qquad = \{m-n : m,n \in \mathbb{N}_0, \text{ there is an open neighbourhood}\\ &\qquad\qquad U \subseteq U_m \cap U_n \text{ of } h^{-1}(y) \text{ such that } \sigma_m(x) = \sigma_n(x) \text{ for all } x \in U\}\end{aligned}$$

for all $y \in Y$.

The following result follows from [Carlsen et al., 2017b, Theorem 8.2 and Proposition 8.3]

Theorem 5.4. *Let $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms. Then the following are equivalent.*

(1) There is a continuous orbit equivalence between $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ that preserves essential stabilisers.

(2) $G(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $G(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ are topological isomorphic.

(3) *There is a $*$-isomorphism $\phi : C_r^*((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})) \to C_r^*((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ such that $\phi(C_0(X)) = C_0(Y)$.*

The following result follows from the previous result and [Steinberg, 2019, Theorem 5.7].

Theorem 5.5. *Let $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are totally disconnected second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms, and let R be an indecomposable unital ring. Then the following are equivalent.*

(1) *There is a continuous orbit equivalence between $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ that preserves essential stabilisers.*

(2) *$G(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $G(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ are topological isomorphic.*

(3) *There is an R-algebra isomorphism $\phi : A_R((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})) \to A_R((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ such that $\phi(A_R(X)) = A_R(Y)$.*

(4) *There is a ring isomorphism $\phi : A_R((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})) \to A_R((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ such that $\phi(A_R(X)) = A_R(Y)$.*

5.3. **Continuous orbit equivalence between graphs.** Let E and F be two countable graphs. A continous orbit equivalence (h,k,l,k',l') between $(\partial E,(\partial E^{\geq n},\sigma_n)_{n\in\mathbb{N}_0})$ and $(\partial F,(\partial F^{\geq n},\sigma_n)_{n\in\mathbb{N}_0})$ is said to *preserve isolated eventually periodic points* if $h : \partial E \to \partial F$ and $h^{-1} : \partial F \to \partial E$ map isolated eventually periodic points to isolated eventually periodic points.

The next result follows from Theorem 5.4, Theorem 5.5, and [Carlsen&Winger, 2018, Theorem 4.2], cf. also [Arklint et al., 2018, Theorem 5.3].

Theorem 5.6. *Let E and F be two graphs and let R be a indecomposable unital commutative ring. Then the following are equivalent.*

(1) *There is a continous orbit equivalence between $(\partial E,(\partial E^{\geq n},\sigma_n)_{n\in\mathbb{N}_0})$ and $(\partial F,(\partial F^{\geq n},\sigma_n)_{n\in\mathbb{N}_0})$ that preserves isolated eventually periodic points.*

(2) *$G(E)$ and $G(F)$ are topological isomorphic.*

(3) *There is a $*$-isomorphism $\phi : C^*(E) \to C^*(F)$ such that $\phi(D(E)) = D(F)$.*

(4) *There is a ring isomorphism $\phi : L_R(E) \to L_R(F)$ such that $\phi(D_R(E)) = D_R(F)$.*

(5) *There is a R-algebra isomorphism $\phi : L_R(E) \to L_R(F)$ such that $\phi(D_R(E)) = D_R(F)$.*

5.4. **Equivariant diagonal-preserving isomorphism of groupoid C^*-algebras and graded isomorphism of groupoids.** The following result is proven in [Carlsen et al., 2017b, Theorem 6.2].

Theorem 5.7. *Let Γ be an abelian discrete group, let G_1 and G_2 be locally compact Hausdorff étale groupoids, let $c_1 : G_1 \to \Gamma$ and $c_2 : G_2 \to \Gamma$ be continous cocycles, and consider the following two conditions.*

(1) There is a topological groupoid isomorphism $\phi : G_1 \to G_2$ such that $c_1 = c_2 \circ \phi$.

(2) There is a $*$-isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\phi(\beta_\zeta^{c_1}(x)) = \beta_\zeta^{c_2}(\phi(x))$ for $x \in C_r^*(G_1)$ and $\zeta \in \widehat{\Gamma}$.

Then $1 \Rightarrow 2$. If moreover G_1 and G_2 are second-countable and each $\mathrm{Iso}(c_i^{-1}(0))^\circ$ is torsion-free and abelian, then $2 \Rightarrow 1$.

The following result is a special case of [Steinberg, 2019, Theorem 5.7].

Theorem 5.8. *Let Γ be an abelian discrete group, let G_1 and G_2 be Hausdorff ample étale groupoids, let $c_1 : G_1 \to \Gamma$ and $c_2 : G_2 \to \Gamma$ be continous cocycles, let R be a unital commutative ring, and consider the following two conditions.*

(1) There is a topological groupoid isomorphism $\phi : G_1 \to G_2$ such that $c_1 = c_2 \circ \phi$.

(2) There is a Γ-graded R-algebra isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.

(3) There is a Γ-graded ring isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.

Then $1 \Rightarrow 2 \Rightarrow 3$. If moreover each $\mathrm{Iso}(c_i^{-1}(0))^\circ$ is free abelian and R is indecomposable, then $3 \Rightarrow 1$.

5.5. **Eventually conjugacy and graded isomorphisms.** A continous orbit equivalence (h, k, l, k', l') between two Deaconu–Renault systems $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ that preserves essential stabilisers is called an *eventually conjugacy* if $l(x) = k(x) + 1$ for all $x \in X$.

The following result is proven in [Carlsen et al., 2017b, Theorem 8.10].

Theorem 5.9. *Let $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms. Then the following are equivalent.*

(1) There is a eventually conjugacy between $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$.

(2) $G(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ *and* $G(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ *are graded topological isomorphic.*

(3) *There is a* $*$*-isomorphism* $\phi: C_r^*((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0}))\to C_r^*((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ *such that* $\phi(C_0(X)) = C_0(Y)$ *and* $\phi(\beta_\gamma(x)) = \beta_\gamma(\phi(x))$ *for* $x\in C_r^*((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0}))$ *and* $\gamma\in\mathbb{T}$.

The following result follows from Theorem 5.9 and [Steinberg, 2019, Theorem 5.7].

Theorem 5.10. *Let* $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ *and* $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ *be two Deaconu–Renault systems where* X *and* Y *are totally disconnected second-countable locally compact Hausdorff space and* $\sigma_m: U_m\to X$ *and* $\tau_m: V_m\to Y$ *are local homeomorphisms, and let* R *be an indecomposable unital ring. Then the following are equivalent.*

(1) *There is a eventually conjugacy between* $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ *and* $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$.

(2) $G(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ *and* $G(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ *are graded topological isomorphic.*

(3) *There is a graded* R*-algebra isomorphism* $\phi: A_R((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0}))\to A_R((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ *such that* $\phi(A_R(X)) = A_R(Y)$.

(4) *There is a graded ring isomorphism* $\phi: A_R((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0}))\to A_R((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ *such that* $\phi(A_R(X)) = A_R(Y)$.

5.6. **Eventually conjugacy between graphs.** The next result follows from Theorem 5.9 and Theorem 5.10, cf. also [Carlsen&Rout, 2017, Theorem 4.1].

Theorem 5.11. *Let* E *and* F *be two countable graphs and let* R *be an indecomposable unital commutative ring. Then the following are equivalent.*

(1) *There is a eventually conjugacy between* $(\partial E,(\partial E^{\geq n},\sigma_n)_{n\in\mathbb{N}_0})$ *and* $(\partial F,(\partial F^{\geq n},\sigma_n)_{n\in\mathbb{N}_0})$.

(2) $G(E)$ *and* $G(F)$ *are graded topological isomorphic.*

(3) *There is a* $*$*-isomorphism* $\phi: C^*(E)\to C^*(F)$ *such that* $\phi(D(E)) = D(F)$ *and* $\phi(\beta_\gamma(x)) = \beta_\gamma(\phi(x))$ *for* $x\in C^*(E)$ *and* $\gamma\in\mathbb{T}$.

(4) *There is a graded ring isomorphism* $\phi: L_R(E)\to L_R(F)$ *such that* $\phi(D_R(E)) = D_R(F)$.

(5) *There is a graded* R*-algebra isomorphism* $\phi: L_R(E)\to L_R(F)$ *such that* $\phi(D_R(E)) = D_R(F)$.

6. Equivalence of groupoids

6.1. **Similar groupoids.**

- Let G and H be Hausdorff étale groupoids and let $\rho, \sigma : G \to H$ be continuous groupoid homomorphisms. We say that ρ and σ are *similar* if there is a continuous map $\theta : G^{(0)} \to H$ such that $\theta(r(\eta))\rho(\eta) = \sigma(\eta)\theta(s(\eta))$ for all $\eta \in G$.
- We say that G and H are *similar* if there are continuous groupoid homomorphisms $\rho : G \to H$ and $\sigma : H \to G$ such that ρ and σ are local homeomorphisms, $\sigma \circ \rho$ is similar to id_G and $\rho \circ \sigma$ is similar to id_H, see [Matui, 2012, Definition 3.4], [Renault, 1980, Definition I.1.3], and [Farsi et al., 2018, Definition 3.1].

6.2. **Equivalent groupoids.**

- Let G be a locally compact Hausdorff groupoid and let Z be a locally compact Hausdorff space. Then Z is *a left G-space* if there is a continuous open map $r : Z \to G^{(0)}$ and a continuous map $(\eta, z) = \eta z$ from $\{(\eta, z) : \eta \in G,\ z \in Z,\ s(\eta) = r(z)\}$ to Z such that $r(\eta z) = r(\eta)$ and $(\eta_1\eta_2)z = \eta_1(\eta_2 z)$ and such that $r(z)z = z$.
- We say that Z is a free and proper left G-space if the map $(\eta, z) \mapsto (\eta z, z)$ is a proper injection from $\{(\eta, z) : \eta \in G,\ z \in Z,\ s(\eta) = r(z)\}$ to $Z \times Z$.

  Definition 6.1. <4->[Cf. [Muhly et al., 1987, Definition 2.1], [Renault, 1982, Section 3], and [Farsi et al., 2018, Definition 3.7].] G and H are *equivalent* if there is a locally compact Hausdorff space Z such that

  (1) Z is a free proper left G-space with fibre map $r : Z \to G^{(0)}$,

  (2) Z is a free proper right H-space with fibre map $s : Z \to H^{(0)}$,

  (3) the actions of G and H on Z commute,

  (4) $r : Z \to G^{(0)}$ induces a homeomorphism $Z/H \to G^{(0)}$,

  (5) $s : Z \to H^{(0)}$ induces a homeomorphism $G\backslash Z \to H^{(0)}$.

6.3. **Ampliations.**

- Let G be a Hausdorff étale groupoid, let X be a locally compact Hausdorff space, and let $\psi : X \to G^{(0)}$ be a surjective local homeomorphism.
- Then the *ampliation* or the *blow-up* of G corresponding to ψ is the Hausdorff étale groupoid

$$G^{\psi} := \{(x, \eta, y) : x, y \in X,\ \eta \in G,\ \psi(x) = r(\eta),\ s(\eta) = \psi(y)\}$$

where
$$(G^{\psi})^{(0)} := \{(x, \psi(x), x) : x \in X,\ \eta \in G^{(0)}\}$$
which we identify with X in the natural way, and $r((x, \eta, y)) = x$, $s((x, \eta, y)) = y$, and $(x, \eta, y)^{-1} = (y, \eta^{-1}, x)$ for $(x, \eta, y) \in G^{\psi}$, $(x, \eta, y)(y, \zeta, z) = (x, \eta\zeta, z)$, and the topology is the relative topology of $X \times G \times X$, see [Farsi et al., 2018] or [Williams, 2016, Section 3.3].

6.4. **Morita equivalence.**

- Let G and H be topological groupoids.
- A continuous groupoid homomorphism $\phi : G \to H$ is a *weak equivalence* if
  (1) the map $(x, \zeta) \mapsto s(\zeta)$ is a surjective local homeomorphism from $\{(x, \zeta) : x \in G^{(0)},\ \zeta \in H,\ \phi(x) = r(\zeta)\}$ to $H^{(0)}$,
  (2) the map $\eta \mapsto (r(\eta), \phi(\eta), s(\eta))$ from G to the ampliation $\{(x, \zeta, y) : x, y \in G^{(0)},\ \zeta \in H,\ r(\zeta) = \phi(x),\ s(\zeta) = \phi(y)\}$ of H with respect to $\phi|_{G^{(0)}}$ is a topological isomorphism.
- G and H are *Morita equivalent* if there is a topological groupoid K, a weak equivalence $\phi : K \to G$, and a weak equivalence $\psi : K \to H$, see [Crainic&Moerdijk, 2000, Section 4.5] or [Farsi et al., 2018, Definition 3.4].

6.5. **Kakutani equivalence.** Recall that a subset $U \subseteq G^{(0)}$ is *full* if $r(GU) = G^{(0)}$.

Definition 6.2 (Cf. [Matui, 2012, Definition 4.1] and [Farsi et al., 2018, Definition 3.8])**.** Two Hausdorff étale groupoids are *weakly Kakutani equivalent* if there are full open subsets $X \subseteq G^{(0)}$ and $Y \subseteq H^{(0)}$ such that $G|_X$ and $H|_Y$ are topological isomorphic. They are *Kakutani equivalent* if X and Y can be chosen to be clopen.

6.6. **Stabilised isomorphism.**

- Let $\mathcal{R}$ be the groupoid of the equivalence relation $\mathbb{N} \times \mathbb{N}$ on $\mathbb{N}$, and equip $\mathcal{R}$ with the discrete topology.
- Then $C^*(\mathcal{R}) \cong \mathcal{K}$ and $A_R(\mathcal{R}) \cong M_\infty(R)$ for any unital commutative ring R.
- It follows that $C^*(G \times \mathcal{R}) \cong C^*(G) \otimes \mathcal{K}$ for any locally compact Hausdorff étale groupoid.
- And that $A_R(G \times \mathcal{R}) \cong A_R(G) \otimes M_\infty(R)$ for any ample étale groupoid G and any unital commutative ring.

6.7. **Equivalence of ample groupoids.** The following result is proved in [Farsi et al., 2018, Theorem 3.12].

Theorem 6.3. *Let G and H be ample Hausdorff étale groupoids with σ-compact unit spaces. Then the following are equivalent.*

(1) G and H are similar.

(2) G and H are equivalent.

(3) G and H admit isomorphic ampliations.

(4) G and H are Morita equivalent.

(5) G and H are Kakutani equivalent.

(6) G and H are weakly Kakutani equivalent.

(7) $G \times \mathcal{R} \cong H \times \mathcal{R}$.

6.8. **Diagonal-preserving stable isomorphism of groupoid C^*-algebras and stable isomorphism of groupoids.** The following result follows from Theorem 5.1 and the discussion above.

Theorem 6.4. *Let G_1 and G_2 be locally compact Hausdorff étale groupoids and consider the following two conditions.*

(1) $G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}$.

(2) There is a $*$-isomorphism $\phi : C^*(G_1) \otimes \mathcal{K} \to C^*(G_2) \otimes \mathcal{K}$ such that $\phi(C_0(G_1^{(0)}) \otimes \mathcal{C}) = C_0(G_2^{(0)}) \otimes \mathcal{C}$.

Then $1 \Rightarrow 2$. If moreover G_1 and G_2 are second-countable and each $\mathrm{Iso}(G_i)^\circ$ is torsion-free and abelian, then $2 \Rightarrow 1$.

6.9. **Diagonal-preserving stable isomorphism of Steinberg algebras and stable isomorphism of groupoids.** The following result follows from Theorem 5.2 and the discussion above.

Theorem 6.5. *Let G_1 and G_2 be Hausdorff ample étale groupoids, let R be a unital commutative ring, and consider the following two conditions.*

(1) $G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}$.

(2) There is a R-algebra isomorphism $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ such that $\phi(A_R(G) \otimes D_\infty(R)) = A_R(H) \otimes D_\infty(R)$.

(3) There is a ring isomorphism $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ such that $\phi(A_R(G) \otimes D_\infty(R)) = A_R(H) \otimes D_\infty(R)$.

Then $1 \Rightarrow 2 \Rightarrow 3$. If moreover each $\mathrm{Iso}(G_i)^\circ$ is free abelian and R is indecomposable, then $3 \Rightarrow 1$.

6.10. **The stabilisation of a graph.**

- If E is a graph, then we denote by SE the graph obtained by attaching a head $\dots e_{3,v}e_{2,v}e_{1,v}$ to every vertex $v \in E^0$, see [Tomforde, 2004, Section 4].
- We equip $G(SE)$ with a $\mathbb{Z}$-grading $(G(SE)_n)_{n\in\mathbb{Z}}$ where $G(SE) = \{(\mu x, |\mu| - |\nu|, \nu x) : n = \#\{\text{edges from } E \text{ in } \mu\} - \#\{\text{edges from } E \text{ in } \mu\}\}$.
- $G(SE)$ is graded isomorphich to $G(E) \times \mathcal{R}$, see [Carlsen et al., 2017a, Lemma 4.1].
- There is a $*$-isomorphism $\phi : C^*(SE) \to C^*(E) \otimes \mathcal{K}$ such that $\phi(\beta'_\gamma(x)) = (\beta_\gamma \otimes \mathrm{id})(\phi(x))$ for $x \in C^*(SE)$ and $\gamma \in \mathbb{T}$.
- $L_R(SE)$ is graded isomorphic to $L_R(E) \otimes M_\infty(R)$ for any unital commutative ring.

6.11. **Shifts of finite type of graphs.**

- Let E be a finite graph with no sinks and no sources.
- Let $X_E = \{(e_n)_{n\in\mathbb{Z}} : e_n \in E^1 \text{ and } r(e_n) = s(e_{n+1}) \text{ for all } n \in \mathbb{Z}\}$.
- Define $\sigma_E : X_E \to X_E$ by $\sigma_E((e_n)_{n\in\mathbb{Z}}) = (f_n)_{n\in\mathbb{Z}}$ where $f_n = e_{n+1}$ for all $n \in \mathbb{Z}$.
- Then (X_E, σ_E) is a shift of finite type.
- If E and F are two finites graph with no sinks and no sources, then (X_E, σ_E) and (X_F, σ_F) are *conjugate* if there is a homeomorphism $\phi : X_E \to X_F$ such that $\phi \circ \sigma_E = \sigma_F \circ \phi$.

6.12. **Graded stable isomorphism for finite graphs.** The following result is shown in [Carlsen&Rout, 2018, Corollary 4.7].

Theorem 6.6. *Let E and F be two finites graphs with no sinks and no sources and let R be a indecomposable unital commutative ring. Then the following are equivalent.*

(1) (X_E, σ_E) and (X_F, σ_F) are conjugate.

(2) $G(SE)$ and $G(SF)$ are graded topological isomorphic.

(3) There is a $*$-isomorphism $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ such that $\phi(D(E) \otimes \mathcal{C}) = D(F) \otimes \mathcal{C}$ and $\phi((\beta_\gamma \otimes \mathrm{id})(x)) = (\beta_\gamma \otimes \mathrm{id})(\phi(x))$ for $x \in C^*(E) \otimes \mathcal{K}$ and $\gamma \in \mathbb{T}$.

(4) There is a graded ring isomorphism $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ such that $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.

(5) *There is a graded R-algebra isomorphism $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ such that $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.*

6.13. **Stable isomorphism for finite graphs.** The following result is shown in [Carlsen et al., 2019, Corollary 6.3].

Theorem 6.7. *Let E and F be two finites graphs with no sinks and no sources and let R be a indecomposable unital commutative ring. Then the following are equivalent.*

(1) *(X_E, σ_E) and (X_F, σ_F) are flow equivalent.*

(2) *There is a continuous orbit equivalence between $(\partial SE, (\partial SE^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$ and $(\partial SF, (\partial SF^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$.*

(3) *$G(SE)$ and $G(SF)$ are topological isomorphic.*

(4) *There is a $*$-isomorphism $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ such that $\phi(D(E) \otimes \mathcal{C}) = D(F) \otimes \mathcal{C}$.*

(5) *There is a ring isomorphism $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ such that $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.*

(6) *There is a R-algebra isomorphism $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ such that $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.*

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